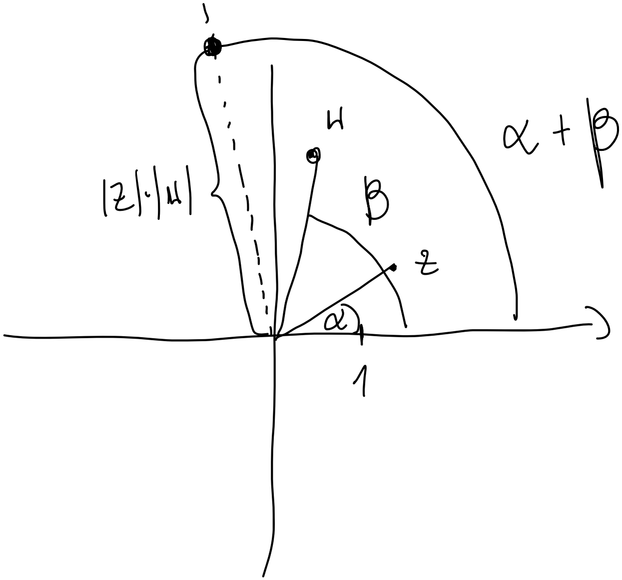
Mnożenie w postaci trygonometrycznej

Jeżeli $z, w \in \mathbb{C}$ oraz

$$z = |z|(\cos\alpha + i\sin\alpha), \qquad w = |w|(\cos\beta + i\sin\beta),$$

to

$$zw = |z||w|(\cos(\alpha+\beta) + i\sin(\alpha+\beta)).$$



Wzór de Moivre'a

Jeżeli $z \in \mathbb{C}$ i
$$z = |z|(\cos\alpha + i\sin\alpha)$$
oraz $n \in \mathbb{N}$, to
$$z^n = |z|^n(\cos(n\alpha) + i\sin(n\alpha)).$$

$$z^2 = z \cdot z = |z| \cdot |z| \left(\cos(\alpha+\alpha) + i\sin(\alpha+\alpha)\right) =$$
$$= |z|^2 (\cos 2\alpha + i\sin 2\alpha)$$

$$\cos(3\alpha) = \dots ?$$

$$(\cos\alpha + i\sin\alpha)^3 = \boxed{\cos 3\alpha + i\sin 3\alpha}$$

$$(\cos\alpha + i\sin\alpha)^3 = \cos^3\alpha + 3\cos^2\alpha\, i\sin\alpha +$$

$$+ 3\cos\alpha \underbrace{i^2}_{=-1}\sin^2\alpha + \underbrace{i^3}_{=i^2\cdot i = -i}\sin^3\alpha =$$

$$= \boxed{\cos^3\alpha - 3\cos\alpha\sin^2\alpha + i\left(3\cos^2\alpha\sin\alpha - \sin^3\alpha\right)}$$

$$\boxed{\begin{aligned} \cos 3\alpha &= \cos^3\alpha - 3\cos\alpha\sin^2\alpha \\ \sin 3\alpha &= 3\cos^2\alpha\sin\alpha - \sin^3\alpha \end{aligned}}$$

$$z = a + bi = |z|(\cos\alpha + i\sin\alpha)$$

Postać wykładnicza

Jeżeli $z \in \mathbb{C}$ i

$$z = |z|(\underbrace{\cos\alpha + i\sin\alpha}),$$

to zapis

$$z = |z|e^{i\alpha}$$

nazywamy **postacią wykładniczą** liczby z.

$$\boxed{e^{i\alpha} = \cos\alpha + i\sin\alpha} \; ?$$

$$f(x) = \sum_{k=0}^{\cancel{n}\; +\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \cancel{R_n}$$

$n \to +\infty$ („dla małych R_n")

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$x \in \mathbb{R}$

$$\boxed{\begin{aligned} i^1 &= i \\ i^2 &= -1 \\ i^3 &= -i \\ i^4 &= 1 \end{aligned}}$$

$$1 + (i\alpha)^1 + (i\alpha)^2 + \ldots = \left(1 - \frac{\alpha^2}{2} + \frac{\alpha^4}{4!} - \ldots\right) + i\left(\alpha - \frac{\alpha^3}{3!} + \frac{\alpha^5}{5!} - \ldots\right)$$

$$e^{i\alpha} = \cos\alpha + i\sin\alpha$$

$$\alpha = \pi = 180^\circ$$

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 + 0$$

$$\boxed{e^{i\pi} + 1 = 0}$$

$$\sqrt{z} = ?$$

$$\sqrt{4} = 2$$

$$a \in \mathbb{R} \quad \sqrt[3]{a} = b \iff b^3 = a$$

$$a \geq 0 \quad \sqrt{a} = b \iff b^2 = a \wedge b \geq 0$$

?

$$f(x) = \sqrt{x}$$

$$\sqrt{4} = \{2, -2\}$$

Pierwiastek zespolony

Jeżeli $z \in \mathbb{C}$ i $n \in \mathbb{N}$ to

$$\sqrt[n]{z} = \{w \in \mathbb{C} \colon w^n = z\}.$$

w $\mathbb{R}$

$$\sqrt{4} = 2$$

w $\mathbb{C}$

$$\sqrt{4} = \{2, -2\}$$

$$z = |z| e^{i\alpha} \quad , \quad w = |w| e^{i\beta}$$

$$\left(|w| e^{i\beta}\right)^n = |z| e^{i\alpha} \Rightarrow |w|^n e^{in\beta} = |z| e^{i\alpha}$$

$$|w|^n = |z|$$

$$|w| = \sqrt[n]{|z|}$$

$$n\beta = \alpha + 2k\pi$$

$$\beta = \frac{\alpha + 2k\pi}{n}$$

Pierwiastek zespolony

Jeżeli $z \in \mathbb{C}$, $z \neq 0$, to $\sqrt[n]{z}$ ma dokładnie n różnych elementów. Dla

$$z = |z|(\cos\alpha + i\sin\alpha)$$

mamy

$$\sqrt[n]{z} = \{z_0, z_1, \ldots, z_{n-1}\},$$

gdzie

$|w|$

$$z_k = \sqrt[n]{|z|}\left(\cos\frac{\alpha + 2k\pi}{n} + i\sin\frac{\alpha + 2k\pi}{n}\right)$$

β

dla $k = 0, 1, \ldots, n-1$.

$k = n \rightsquigarrow k = 1$

$$\boxed{x^2 + x + 2 = 0}$$

$$\Delta = -7$$

$$\sqrt{\Delta} = \sqrt{-7} = \{\sqrt{7}\,i, -\sqrt{7}\,i\}$$

$$i^2 = -1$$

$$x_1 = \frac{-1 + \sqrt{7}\,i}{2}$$

$$x_2 = \frac{-1 - \sqrt{7}\,i}{2}$$

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \,\Big|\, \subset \; ?$$

$$x^{100} - \pi x^{89} + i x^{15} - 1 = 0$$

Zasadnicze twierdzenie algebry

Każdy wielomian stopnia $\geqslant 1$ ma pierwiastek zespolony.

Zasadnicze twierdzenie algebry

Każdy wielomian p stopnia $\geqslant 1$ ma dokładnie n pierwiastków zespolonych $z_1, z_2, \ldots, z_n$ i

$$p(z) = a_n(z - z_1)(z - z_2)\ldots(z - z_n).$$

$\mathbb{H}$