1. $\int (\boxed{3x-2})^5 \boxed{dx} = \cancel{\int t^5 dx}$

t

$$= \left| \begin{array}{l} t = 3x - 2 \quad | ()' \\ (t)' = (3x-2)' \\ 1dt = 3\,dx \\ \frac{1}{3}dt = dx \end{array} \right|$$

$$= \int t^5 \frac{1}{3} dt = \frac{1}{3} \int t^5 dt = \frac{1}{3} \frac{t^6}{6} + C =$$

$$= \frac{1}{18}(3x-2)^6 + C$$

---

2. $\int \frac{1}{\sqrt{2-x^2}} dx = \left| \begin{array}{l} x = \sqrt{2}\, t \\ dx = \sqrt{2}\, dt \end{array} \right| =$

$$\boxed{\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x + C}$$

$$= \int \frac{1}{\sqrt{2-2t^2}} \sqrt{2}\, dt = \cancel{\sqrt{2}} \int \frac{1}{\cancel{\sqrt{2}}\sqrt{1-t^2}} dt =$$

$$= \arcsin t + C = \arcsin \frac{x}{\sqrt{2}} + C$$

---

3. $\int \frac{e^{1/x}}{x^2} dx = \left| \begin{array}{l} t = \boxed{f(x)} \\ dt = \boxed{f'(x)dx} \end{array} \right| = \cancel{\left| \begin{array}{l} t = x^2 \\ dt = 2\boxed{x\,dx} \end{array} \right|}$

f f'

$$= \left| \begin{array}{l} t = \frac{1}{x} \\ dt = -\frac{1}{x^2}dx \end{array} \right| = -\int e^t dt = -e^t + C =$$

$$= \boxed{-e^{1/x} + C}$$

$$\int e^{1/x} dx, \quad \int e^{-x^2} dx, \quad \int \frac{\sin x}{x} dx$$

4. $\int \operatorname{arctg} x \, dx = \int (x)' \operatorname{arctg} x \, dx =$

$$= x \operatorname{arctg} x - \underbrace{\int \frac{x}{1+x^2} dx}$$

$$\int \frac{x}{1+x^2} dx = \left| \begin{array}{l} t = 1 + x^2 \\ dt = 2x \, dx \\ \frac{1}{2} dt = x \, dx \end{array} \right| =$$

$$= \frac{1}{2} \int \frac{dt}{t} = \frac{1}{2} \ln |t| + C = \frac{1}{2} \ln |1+x^2| + C =$$

$$= \frac{1}{2} \ln (1+x^2) + C = \ln \sqrt{1+x^2} + C$$

$a \in \mathbb{R}$

5.

$$\frac{A}{ax+b}$$

$$\int \frac{dx}{x-a} = \left| \begin{array}{l} t = x - a \\ dt = dx \end{array} \right| = \int \frac{dt}{t} =$$

$$= \ln |t| + C = \ln |x-a| + C$$

$n \geq 2$

$$\int \frac{dx}{(x-a)^n} = \int \frac{dt}{t^n} = \int t^{-n} dt = \frac{t^{-n+1}}{-n+1} + C$$

$$= \frac{-1}{n-1} \cdot \frac{1}{t^{n-1}} + C = \frac{-1}{n-1} \cdot \frac{1}{(x-a)^{n-1}} + C$$

6. $\displaystyle\int \frac{Ax+B}{ax^2+bx+c}\,dx \qquad \Delta = b^2-4ac < 0$

$$\int \frac{x}{x^2+3x+4}\,dx = \left\{ (x^2+3x+4)' = 2x+3 \right\}$$

$$= \int \frac{\frac{1}{2}(2x+3)-\frac{3}{2}}{x^2+3x+4}\,dx = \frac{1}{2}\underbrace{\int \frac{2x+3}{x^2+3x+4}\,dx}_{I_1} - \frac{3}{2}\underbrace{\int \frac{dx}{x^2+3x+4}}_{I_2}$$

$$\boxed{I_1} = \left\{ \begin{array}{l} t = x^2+3x+4 \\ dt = (2x+3)\,dx \end{array} \right\} = \int \frac{1}{t}\,dt = \ln|t| + C =$$

$$= \ln|x^2+3x+4| + C = \boxed{\ln(x^2+3x+4) + C}$$

$$\boxed{I_2} = \int \frac{dx}{x^2+3x+4} = \int \frac{dx}{(x+3/2)^2+\frac{7}{4}} = \left| \begin{array}{l} t = x+\frac{3}{2} \\ dt = dx \end{array} \right|$$

$$= \int \frac{dt}{t^2+\frac{7}{4}} = \left| \begin{array}{l} t = \sqrt{\frac{7}{4}}\,s \\ dt = \sqrt{\frac{7}{4}}\,ds \end{array} \right| =$$

$$= \int \frac{\sqrt{\frac{7}{4}}\,ds}{\frac{7}{4}s^2+\frac{7}{4}} = \frac{\sqrt{\frac{7}{4}}}{\frac{7}{4}} \int \frac{ds}{s^2+1} = \frac{2}{\sqrt{7}}\,\mathrm{arctg}\, s + C =$$

$$= \frac{2}{\sqrt{7}}\,\mathrm{arctg}\left(\frac{2}{\sqrt{7}}\,t\right) + C = \boxed{\frac{2}{\sqrt{7}}\,\mathrm{arctg}\left(\frac{2}{\sqrt{7}}\left(x+\frac{3}{2}\right)\right) + C}$$

$$\int \frac{Ax+B}{ax^2+bx+c} = \square \ln(\quad) + \square\,\mathrm{arctg}(\quad)$$

$\Delta < 0$

7. $\int \frac{dx}{(x^2+1)^2} = \int \frac{\boxed{1+x^2-x^2}}{(x^2+1)^2}\,dx =$

$= \underbrace{\int \frac{x^2+1}{(x^2+1)^2}\,dx}_{I_1} - \underbrace{\int \frac{x^2}{(x^2+1)^2}\,dx}_{I_2}$

$I_1 = \int \frac{dx}{x^2+1} = \operatorname{arctg} x + C$

$\boxed{I_2} = \int \frac{x^2}{(x^2+1)^2}\,dx = \int x \cdot \boxed{\frac{x}{(x^2+1)^2}}\,dx =$

$\int \frac{x}{(x^2+1)^2}\,dx = \left| \begin{array}{l} t = x^2+1 \\ dt = 2x\,dx \end{array} \right| = \int \frac{\frac{1}{2}\,dt}{t^2} = -\frac{1}{2}\cdot\frac{1}{t} + C$

$= \frac{-1}{2(x^2+1)} + C$

$= \int x \cdot \boxed{\left(\frac{-1}{2(x^2+1)}\right)'}\,dx = x \cdot \frac{-1}{2(x^2+1)} + \int \frac{1}{2(x^2+1)}(x)'\,dx =$

$= -\frac{x}{2(x^2+1)} + \frac{1}{2}\int \frac{1}{x^2+1}\,dx = \boxed{-\frac{x}{2(x^2+1)} + \frac{1}{2}\operatorname{arctg} x + C}$

$\underbrace{\int \frac{dx}{(x^2+1)^n}}_{I_n} = \ ?$

$I_n = \ldots I_{n-1} + \ldots$

$$\int_a^b f(x)\,dx \overset{N-L}{=} F(x)\Big|_a^b = F(b) - F(a)$$

$$\boxed{\int f(x)\,dx = F(x)}$$

---

1. $\displaystyle\int_1^2 x^3 \ln x\,dx$

$$\int x^3 \ln x\,dx = \int \left(\frac{x^4}{4}\right)' \ln x\,dx =$$

$$= \frac{x^4}{4}\ln x - \int \frac{x^4}{4}\cdot\frac{1}{x}\,dx = \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3\,dx =$$

$$= \frac{x^4}{4}\ln x - \frac{1}{4}\cdot\frac{x^4}{4} + C = \frac{x^4}{4}\ln x - \frac{1}{16}x^4 + C$$

$$\int_1^2 x^3 \ln x\,dx = \left(\frac{x^4}{4}\ln x - \frac{1}{16}x^4\right)\Bigg|_1^2 =$$

$$= (4\ln 2 - 1) - \left(0 - \frac{1}{16}\right) =$$

$$= \boxed{4\ln 2 - \frac{15}{16}}$$

Całkowanie przez części

Jeżeli funkcje f i g mają ciągłe pochodne na przedziale $[a, b]$, to

$$\int_a^b f'g = \cancel{f\Big|_a^b + \int_a^b fg'}$$

$$fg\Big|_a^b - \int_a^b fg'$$

$$\int_0^1 xe^{-x}\,dx = \int_0^1 x\left(-e^{-x}\right)'dx =$$

$$= x\left(-e^{-x}\right)\Big|_0^1 - \int_0^1 (x)'\left(-e^{-x}\right)dx =$$

$$= \left(1\cdot\left(-e^{-1}\right) - 0\right) + \int_0^1 e^{-x}\,dx = \frac{-1}{e} - e^{-x}\Big|_0^1 =$$

$$= -\frac{1}{e} - \left(e^{-1} - e^{0}\right) = e^{0} = \boxed{1}$$

Całkowanie przez podstawienie

Jeżeli

- funkcja g jest określona i ciągła w przedziale $[\alpha, \beta]$,
- funkcja g ma ciągłą pochodną w przedziale $[\alpha, \beta]$,
- funkcja f jest ciągła na zbiorze wartości funkcji g,

to dla $\boxed{a = g(\alpha)}$ i $\boxed{b = g(\beta)}$ mamy

$$\boxed{\int_{\alpha}^{\beta}} f(g(x))g'(x)\,dx = \boxed{\int_{a}^{b}} f(t)dt.$$

1. $\int_2^4 \frac{1}{x \ln^2 x}\, dx = \left| \begin{array}{l} t = \ln x \\ dt = \frac{1}{x} dx \end{array} \quad \begin{array}{c|c|c} x & 2 & 4 \\ \hline t & \ln 2 & \ln 4 \end{array} \right|$

$$= \int_{\ln 2}^{\ln 4} \frac{1}{t^2}\, dt = \left. \frac{t^{-1}}{-1} \right|_{\ln 2}^{\ln 4} = -\frac{1}{\ln 4} - \left(-\frac{1}{\ln 2}\right) =$$

$$= \frac{1}{\ln 2} - \frac{1}{\ln 4}$$

2. $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx = \left| \begin{array}{l} x = \pi - t \\ dx = -dt \end{array} \quad \begin{array}{c|c|c} x & 0 & \pi \\ \hline t & \pi & 0 \end{array} \right|$

$$= -\int_{\pi}^{0} \frac{(\pi - t)\sin(\pi - t)}{1 + \cos^2(\pi - t)}\, dt = \int_0^{\pi} \dots\, dt$$

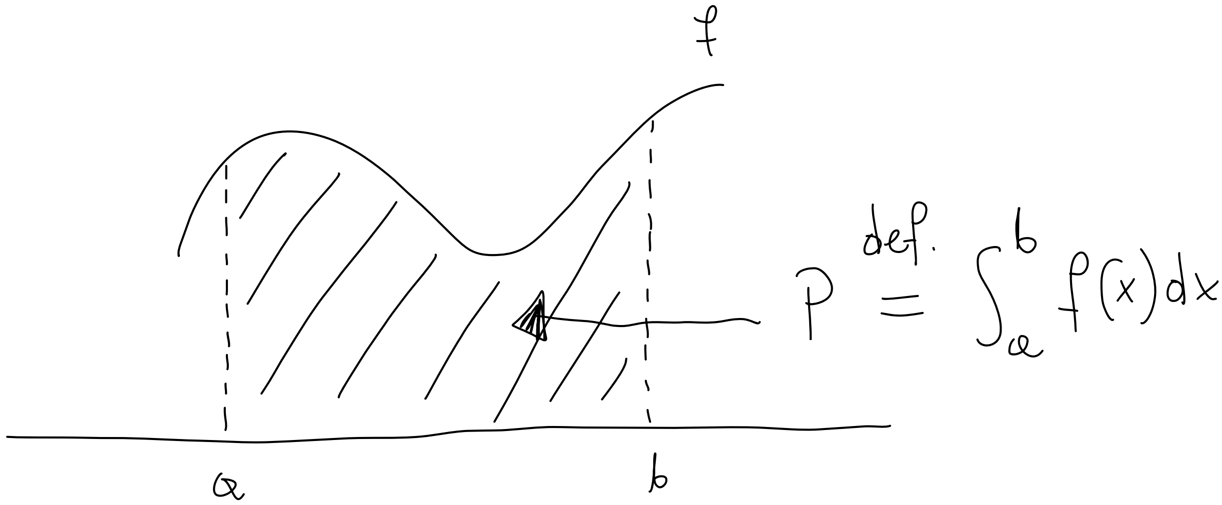
Całka jako pole

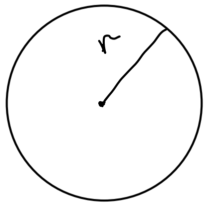
Jeżeli funkcja f jest ciągła i nieujemna na przedziale $[a, b]$, to polem obszaru

$$\{(x, y) \in \mathbb{R}^2 : x \in [a, b],\ 0 \leqslant y \leqslant f(x)\}$$

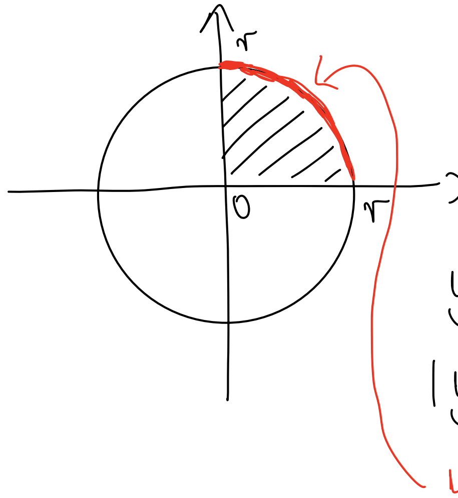
nazywamy całkę

$$\int_a^b f(x)\,dx.$$





$$P = \pi r^2$$



$$x^2 + y^2 = r^2$$

$$y^2 = r^2 - x^2$$

$$|y| = \sqrt{r^2 - x^2}$$

$$y = \sqrt{r^2 - x^2}$$

$$\frac{1}{4}P = \int_0^r \sqrt{\boxed{r^2 - x^2}}\,dx = \left| \begin{array}{l} x = r\sin t \\ dx = r\cos t\,dt \end{array} \quad \begin{array}{c|c|c} x & 0 & r \\ \hline t & 0 & \pi/2 \end{array} \right|$$

$$= \int_0^{\pi/2} \sqrt{r^2 - r^2\sin^2 t}\; r\cos t\,dt =$$

$$= \int_0^{\pi/2} r\sqrt{1 - \sin^2 t}\; r\cos t\,dt = r^2 \int_0^{\pi/2} \sqrt{\cos^2 t}\,\cos t\,dt =$$

$\cos t \geq 0$

$t \in [0, \frac{\pi}{2}]$

$$= r^2 \int_0^{\pi/2} \cos t \cdot \cos t\,dt = r^2 \int_0^{\pi/2} \cos^2 t\,dt =$$

$$= r^2 \int_0^{\pi/2} \frac{1 + \cos 2t}{2}\,dt = \frac{r^2}{2}\left[\int_0^{\pi/2} dt + \int_0^{\pi/2} \cos(2t)\,dt\right] =$$

$$= \frac{r^2}{2}\left[\frac{\pi}{2} + \frac{1}{2}\sin 2t \Big|_0^{\pi/2}\right] =$$

$$= \frac{r^2}{2}\left[\frac{\pi}{2} + \frac{1}{2}(\sin \pi - \sin 0)\right] =$$

$$= \frac{r^2}{2}\left[\frac{\pi}{2} + 0\right] = \frac{r^2}{4}\cdot\pi$$

$$P = 4 \cdot \frac{r^2}{4} \cdot \pi = \boxed{\pi r^2}$$

Całka jako pole

Jeżeli funkcje f i g są ciągłe na przedziale $[a, b]$ oraz

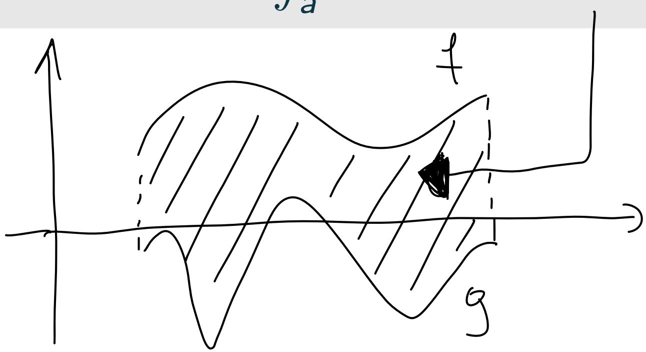
$$g(x) \leqslant f(x), \qquad x \in [a, b],$$

to polem obszaru

$$\{(x, y) \in \mathbb{R}^2 : x \in [a, b],\ g(x) \leqslant y \leqslant f(x)\}$$

nazywamy całkę

$$\int_a^b (f(x) - g(x))\, dx.$$



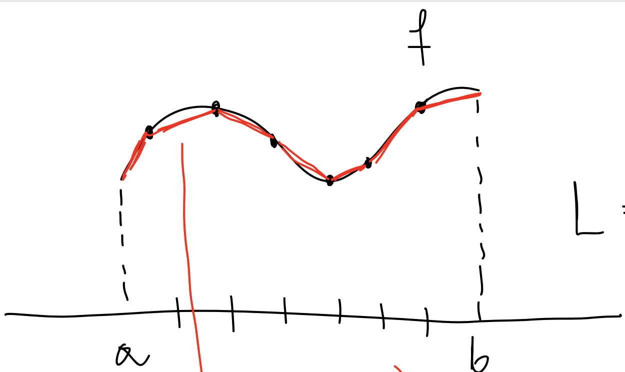
Długość krzywej

Jeżeli funkcja f ma ciągłą pochodną na przedziale $[a, b]$, to długością krzywej

$$\{(x, y) \in \mathbb{R}^2 \colon x \in [a, b],\ \ y = f(x)\}$$

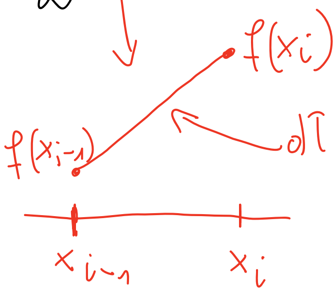
nazywamy całkę

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$



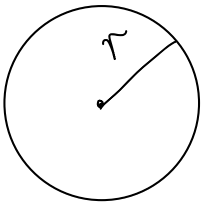
$$L = \sum_{i=1}^{n} (x_i - x_{i-1}) \sqrt{1 + (f'(t_i))^2}$$

$$t_i \in [x_{i-1}, x_i]$$



$$dl = \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} =$$

$$= (x_i - x_{i-1}) \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2}$$



$L = 2\pi r \qquad y = f(x) = \sqrt{r^2 - x^2}$

$$\frac{1}{4} L = \int_0^r \sqrt{1 + (f'(x))^2}\, dx =$$

$$\left\{ f'(x) = \frac{1}{2\sqrt{r^2 - x^2}} \cdot (-2x) = \frac{-x}{\sqrt{r^2 - x^2}} \right\}$$

$$= \int_0^r \sqrt{1 + \left(\frac{-x}{\sqrt{r^2 - x^2}}\right)^2}\, dx = \int_0^r \sqrt{1 + \frac{x^2}{r^2 - x^2}}\, dx =$$

$$= \int_0^r \sqrt{\frac{r^2 - \cancel{x^2} + \cancel{x^2}}{r^2 - x^2}}\, dx = \int_0^r \frac{r}{\sqrt{r^2 - x^2}}\, dx =$$

$$= \left| \begin{array}{l} x = rt \\ dx = r\,dt \end{array} \quad \begin{array}{c|c|c} x & 0 & r \\ \hline t & 0 & 1 \end{array} \right| \quad \int_0^1 \frac{r}{\sqrt{r^2 - r^2 t^2}}\, r\,dt =$$

$$= r^{\cancel{2}} \int_0^1 \frac{dt}{\cancel{r}\sqrt{1 - t^2}} = r \int_0^1 \frac{dt}{\sqrt{1 - t^2}} =$$

$$= r \arcsin t \Big|_0^1 = r\left[\frac{\pi}{2} - 0\right] = \pi \frac{r}{2}$$

$$L = 4 \cdot \pi \frac{r}{2} = \boxed{2\pi r}$$

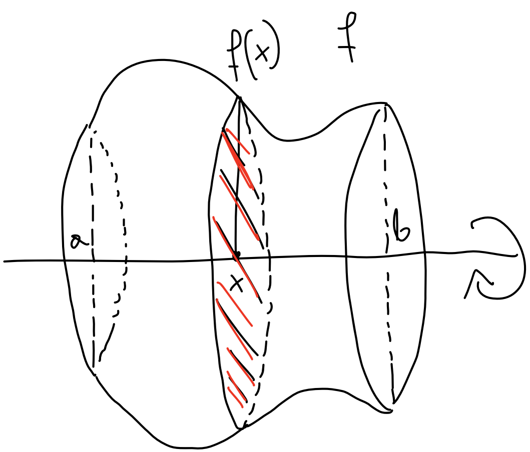
Objętość bryły obrotowej

Jeżeli funkcja f jest ciągła i nieujemna na przedziale $[a, b]$, to objętością bryły obrotowej

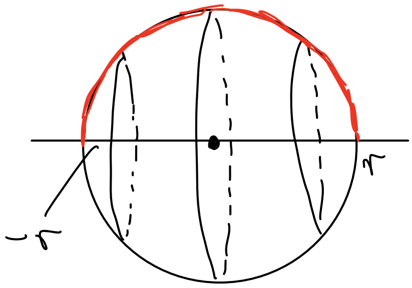
$$\{(x, y, z) \in \mathbb{R}^3 \colon x \in [a, b],\ y^2 + z^2 \leqslant f^2(x)\}$$

nazywamy liczbę

$$\pi \int_a^b f^2(x)\, dx.$$



$$\boxed{\pi (f(x))^2}$$



$y = \sqrt{r^2 - x^2}$

$$V = \pi \int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx = \pi \int_{-r}^{r} \left(r^2 - x^2\right) dx =$$

$$= \pi \left[r^2 \int_{-r}^{r} dx - \int_{-r}^{r} x^2 dx \right] =$$

$$= \pi \left[r^2 \cdot 2r - \frac{x^3}{3} \Big|_{-r}^{r} \right] =$$

$$= \pi \left[2r^3 - \left(\frac{r^3}{3} - \frac{(-r)^3}{3} \right) \right] =$$

$$= \pi \left[2r^3 - \frac{2}{3} r^3 \right] = \pi \frac{4}{3} r^3 = \boxed{\frac{4}{3} \pi r^3}$$

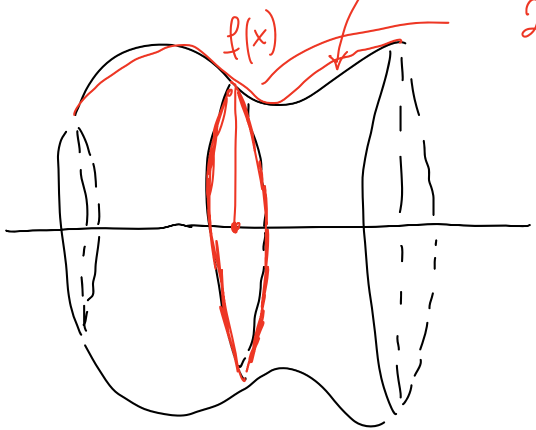
Pole powierzchni bryły obrotowej

Jeżeli funkcja f jest ciągła i nieujemna na przedziale $[a, b]$, to polem powierzchni bocznej bryły obrotowej

$$\{(x, y, z) \in \mathbb{R}^3 \colon x \in [a, b],\ y^2 + z^2 \leqslant f^2(x)\}$$

nazywamy liczbę

$$2\pi \int_a^b f(x) \boxed{\sqrt{1 + (f'(x))^2}}\, dx.$$



$2\pi r$

$2\pi f(x)$