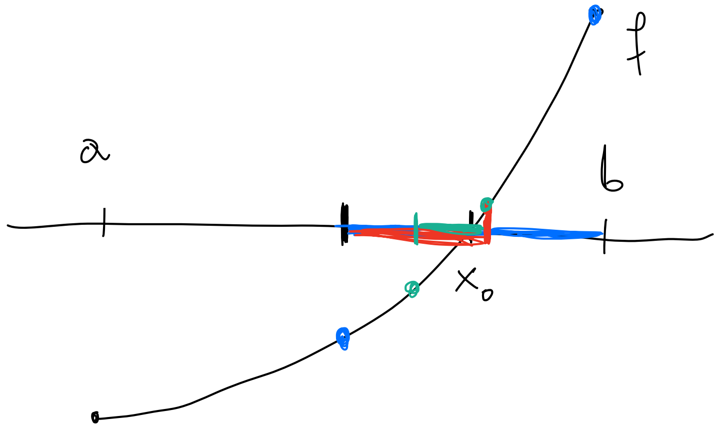
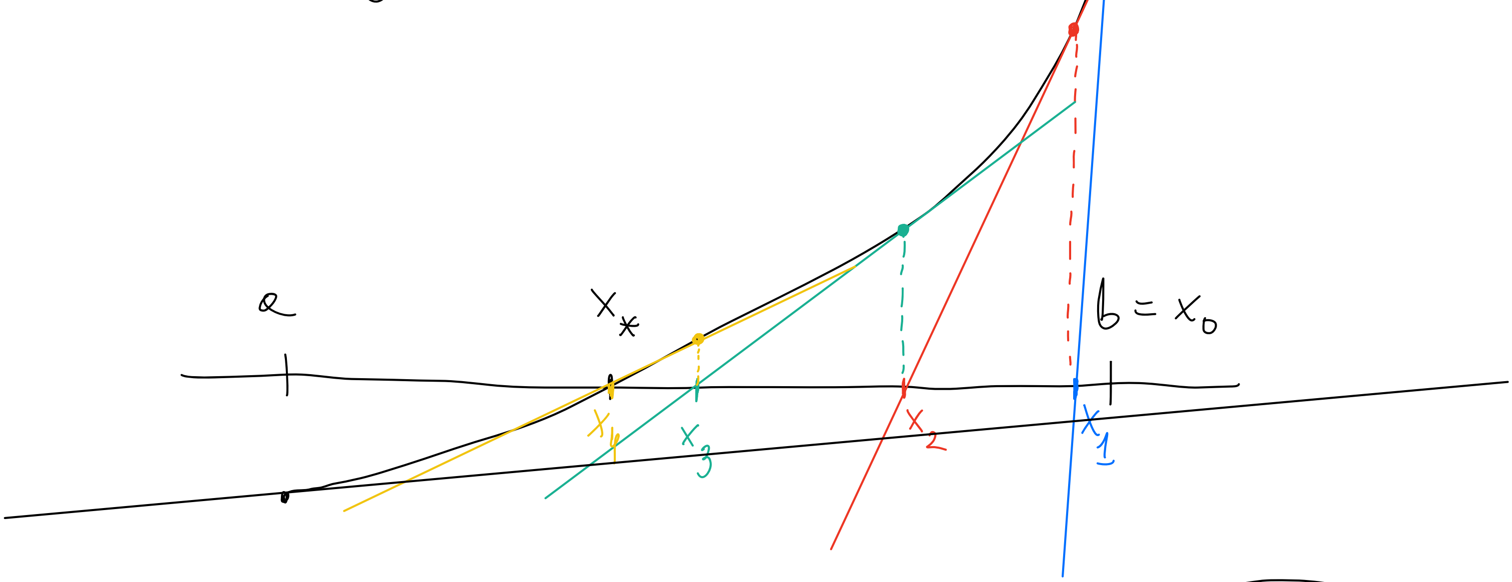


$f(x_0)=0$

$x_0 = ?$

Metoda stycznych Newtona



Wzór Taylora w $x = b$, $n = 1$.

$$f(x) = \underbrace{f(b) + f'(b)(x-b)}_{\text{wzór na styczną w pkt. } b.} + \frac{f''(c)}{2}(x-b)^2$$

Dla $x = x_*$:

$$0 = f(x_*) = f(b) + f'(b)(x_* - b) + \frac{f''(c)}{2}(x_* - b)^2 = 0$$

$$f'(b)\,x_* = -f(b) + f'(b)\,b - \frac{f''(c)}{2}(x_* - b)^2 \quad \Big| : f'(b)$$

$$\boxed{x_* = b - \frac{f(b)}{f'(b)} - \underbrace{\frac{1}{2}\frac{f''(c)}{f'(b)}(x_* - b)^2}_{\text{małe}}}$$

$$x_1 = b - \frac{f(b)}{f'(b)}$$

$$x_* = b - \frac{f(b)}{f'(b)} - \frac{1}{2}\frac{f''(c)}{f'(b)}(x_* - b)^2$$

$$x_1 = b - \frac{f(b)}{f'(b)} \qquad x_* \approx b - \frac{f(b)}{f'(b)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)} \qquad x_* \approx x_2 - \frac{f(x_2)}{f'(x_2)}$$

$\vdots$

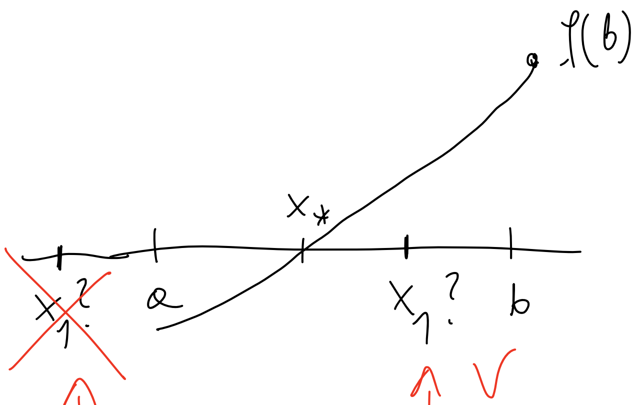
Zał. f' i f'' mają ustalone znaki na $[a,b]$

$\boxed{\text{I} \quad f', f'' > 0}$

II $f' > 0,\ f'' < 0$

III $f' < 0,\ f'' > 0$

IV $f' < 0,\ f'' < 0$



(I) $f' > 0 \Rightarrow f(b) > 0$

$$f'(b) > 0 \Rightarrow \frac{f(b)}{f'(b)} > 0$$

$$x_1 = b - \underbrace{\frac{f(b)}{f'(b)}}_{>0} < b$$

$$x_* = \underbrace{b - \frac{f(b)}{f'(b)}} - \frac{1}{2}\frac{f''(c)}{f'(b)}(x_* - b)^2, \qquad x_1 = \underbrace{b - \frac{f(b)}{f'(b)}}$$

$$x_1 - x_* = \frac{1}{2}\underbrace{\boxed{\frac{f''(c)}{f'(b)}}}_{>0}\underbrace{\boxed{(x_* - b)^2}}_{\geq 0} \geq 0 \Rightarrow x_1 \geq x_*$$

$$\boxed{x_* \leq x_1 < b}$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n \geq 0, \quad \boxed{x_0 = b}$$

$(x_n)_{n\in\mathbb{N}}$ jest malejący i ograniczony z dołu przez x_* $\{ x_* \leq x_{n+1} < x_n \}$

$\Rightarrow$ Istnieje $\underbrace{\text{granica}}_{g}$ ciągu $(x_n)_{n\in\mathbb{N}}$.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \quad \Big| \lim_{n\to+\infty}$$

$$\cancel{g} = \cancel{g} - \frac{f(g)}{f'(g)} \Rightarrow \frac{f(g)}{f'(g)} = 0 \Rightarrow f(g) = 0$$

$$\Rightarrow \boxed{g = x_*}$$

$$\boxed{x_{n+1} - x_* = \frac{1}{2}\frac{f''(c)}{f'(x_n)}(x_* - x_n)^2}$$

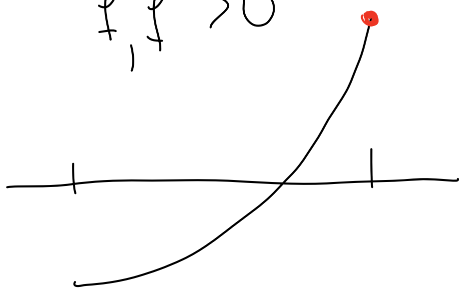
$$|x_{n+1} - x_*| = \frac{1}{2}\boxed{\frac{|f''(c)|}{|f'(x_n)|}}(x_* - x_n)^2$$

$|x_n - x_*| < \frac{1}{100}$

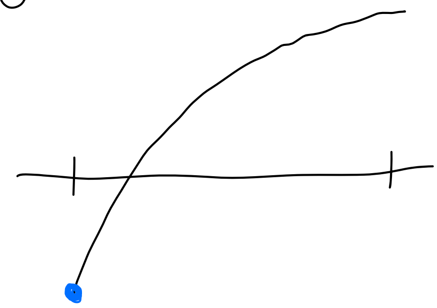
$|x_n - x_*|^2 < \frac{1}{10000}$

$$|x_{n+1} - x_*| \leq \frac{1}{2}\boxed{\frac{\max_x |f''(x)|}{\min_x |f'(x)|}}(x_* - x_n)^2$$

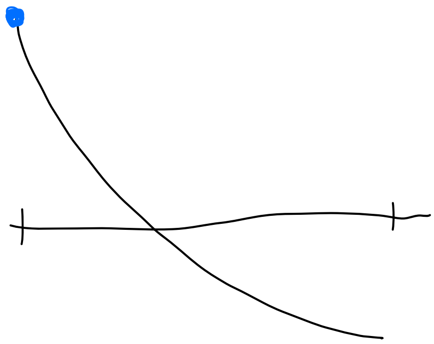
I $f', f'' > 0$



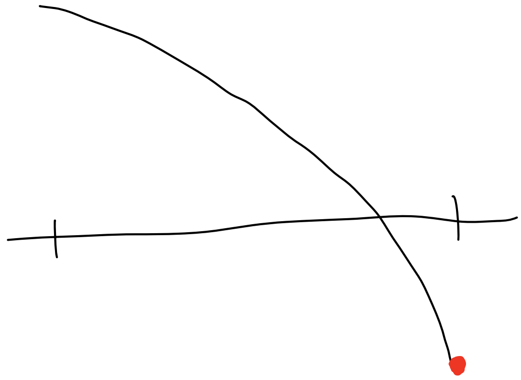
II $f' > 0, f'' < 0$



III $f' < 0, f'' > 0$



IV $f' < 0, f'' < 0$



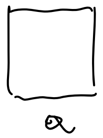
Quake III fast inverse square root

$$\frac{1}{\sqrt{x}}$$

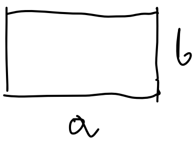
{ a = 1/sqrt(x)

$$\frac{v}{\|v\|}$$

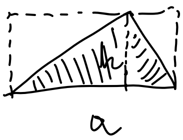
Pole?



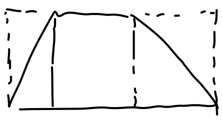
$P = a^2$



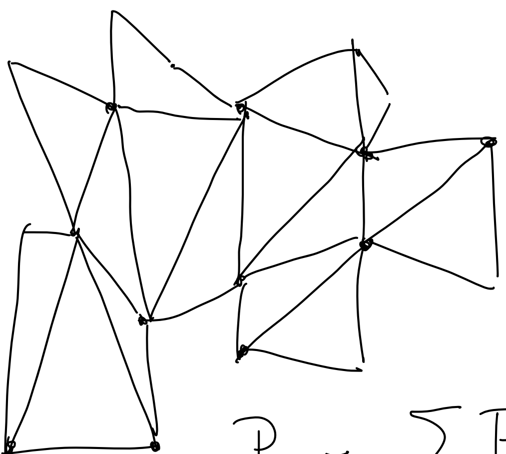
$P = a \cdot b$



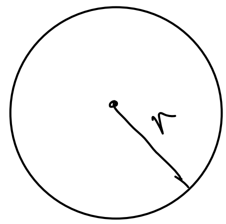
$P = \frac{1}{2} a h$



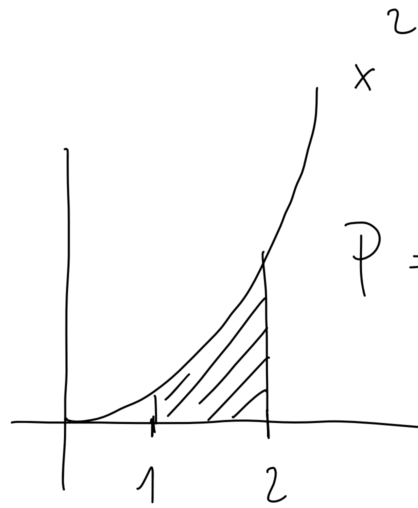
$P = \ldots$



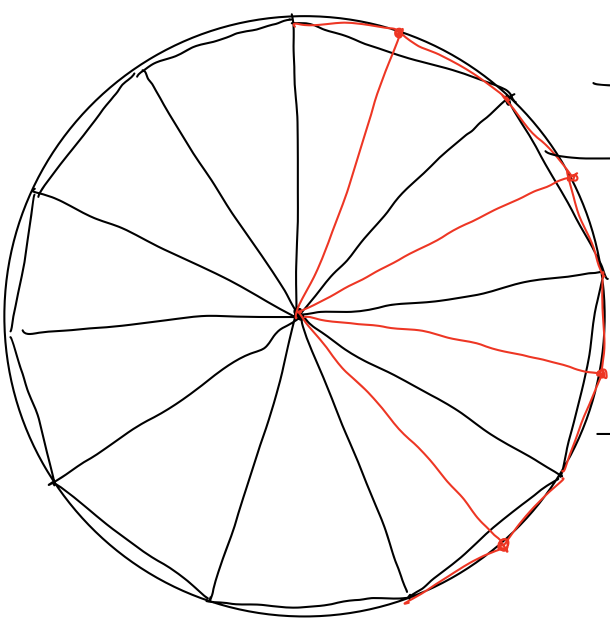
$P = \sum P_{\triangle}$

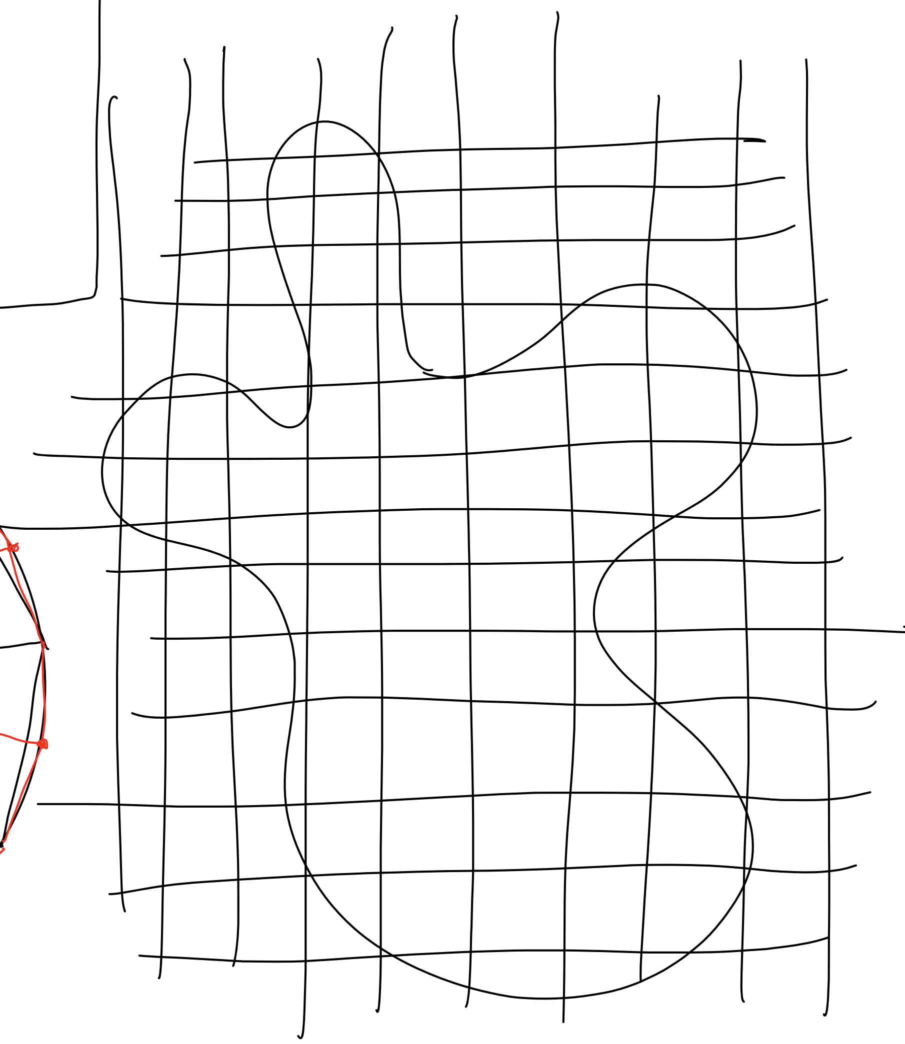


$P = \pi r^2$



$P = ?$

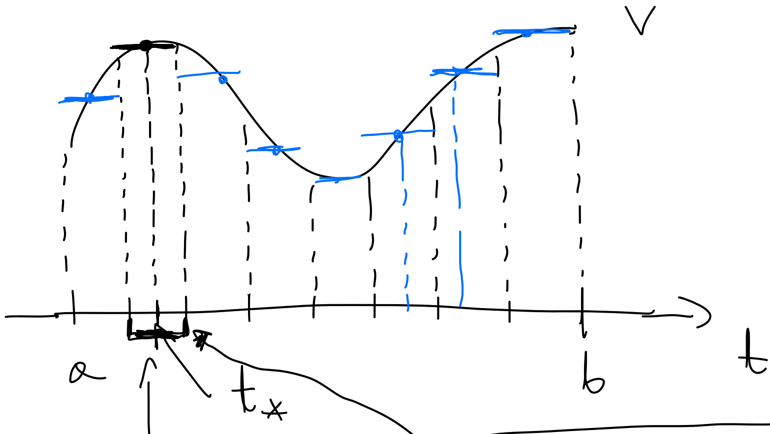




Droga

$$v_{śr} = \frac{s}{t}$$

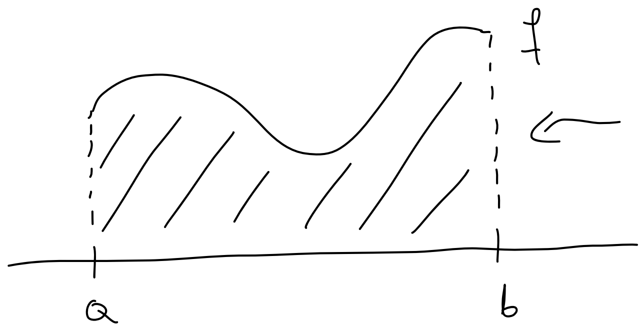
$$\boxed{s = v_{śr} \cdot t}$$



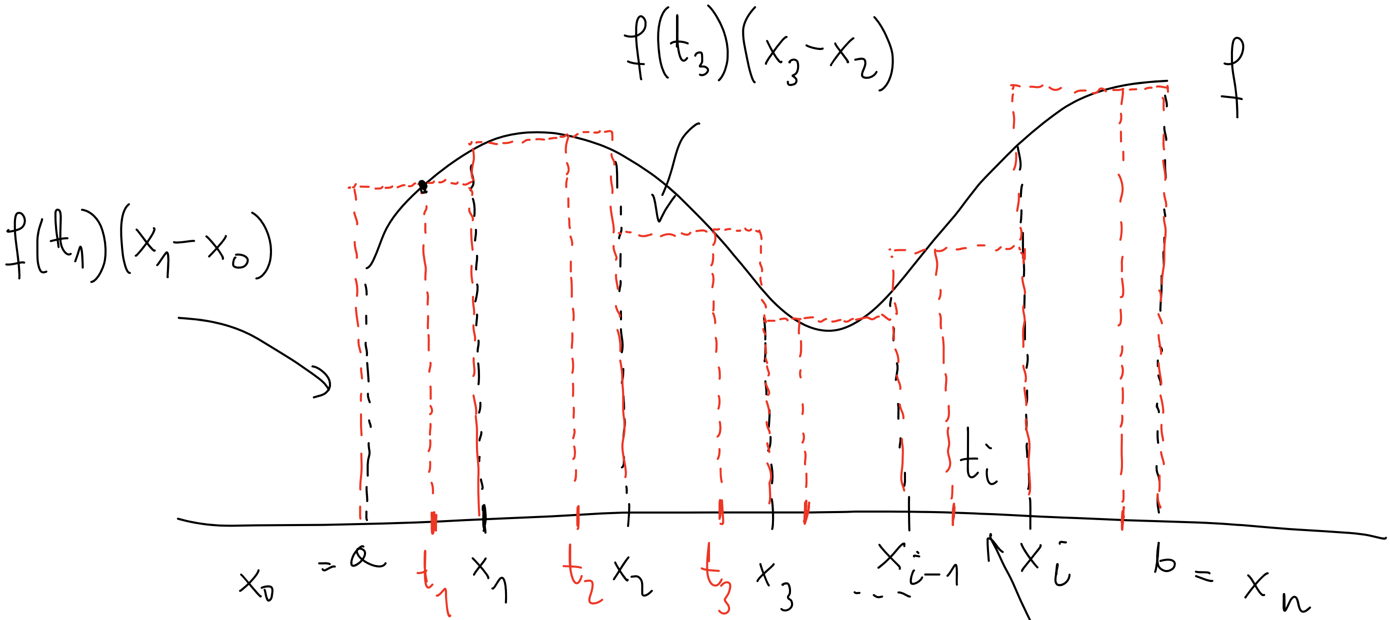
$s \approx v(t_*) \cdot$ długość odcinka czasu

$$s = \sum v(t_*) \cdot d\tau.$$

$$f : [a, b] \to \mathbb{R}$$



$\leftarrow P = ?$



$$a = x_0 < x_1 < \dots < x_n = b$$

$f(t_i)(x_i - x_{i-1})$

suma $P(\ )$

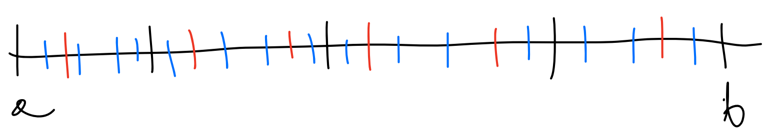
$$\sigma = \boxed{\sum_{i=1}^{n} f(t_i)(x_i - x_{i-1})}$$

$P = \{x_0, x_1, \dots, x_n\}$ ← Podział, punkty podziału

$T = \{t_1, t_2, \dots, t_n\}$ ← zbiór punktów pośrednich

$\sigma = \sigma(f, P, T)$ ← suma całkowa

$$\sigma(f, P_n, T_n) \xrightarrow{n \to +\infty} \int_a^b f$$



Całka Riemanna

Jeżeli funkcja $f : [a, b] \to \mathbb{R}$ jest ograniczona oraz dla dowolnego ciągu normalnego podziałów $(P_n)_{n \in \mathbb{N}}$ przy dowolnym wyborze punktów pośrednich T_n ciąg sum

coraz gęstszy

$$\sigma_n = \sigma_n(f, P_n, T_n)$$

jest zbieżny do tej samej granicy, to mówimy, że funkcja f jest **całkowalna** w sensie Riemanna na przedziale $[a, b]$. Fakt ten zapisujemy w postaci

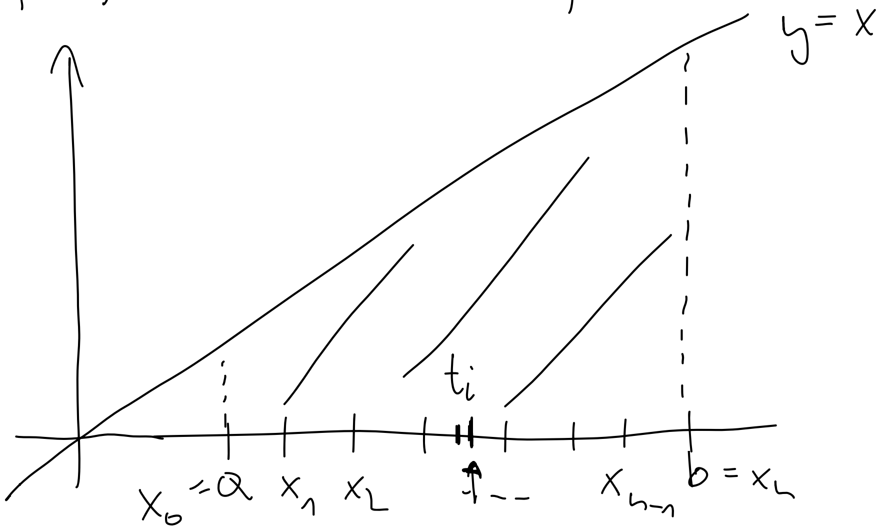
$$\boxed{f \in \mathcal{R}[a, b],}$$

a wartość wspólnej granicy oznaczamy przez

$$\boxed{\int_a^b f(x)\,dx} \quad \text{lub} \quad \boxed{\int_a^b f}$$

$$f(t_i) \quad \boxed{(x_i - x_{i-1})}$$

1. $f(x) = x$ $\qquad f:[a,b] \to \mathbb{R}$



$$\sigma = \sum_{i=1}^{n} f(t_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} t_i (x_i - x_{i-1})$$

$$\tilde{\sigma} = \sum_{i=1}^{n} f\left(\frac{x_i + x_{i-1}}{2}\right)(x_i - x_{i-1}) =$$

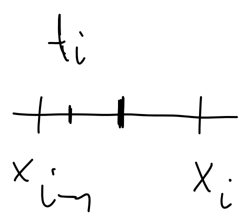
↑ sredek odseka $[x_{i-1}, x_i]$

$$= \sum_{i=1}^{n} \frac{x_i + x_{i-1}}{2}(x_i - x_{i-1}) = \frac{1}{2}\sum_{i=1}^{n}(x_i^2 - x_{i-1}^2)$$

$$= \frac{1}{2}\left[(\cancel{x_1^2} - x_0^2) + (\cancel{x_2^2} - \cancel{x_1^2}) + (\cancel{x_3^2} - \cancel{x_2^2}) + \dots + (x_n^2 - \cancel{x_{n-1}^2})\right]$$

$$= \frac{1}{2}\left(-x_0^2 + x_n^2\right) = \frac{1}{2}(b^2 - a^2)$$

$$\sigma - \tilde{\sigma} = \sum_{i=1}^{n} \underbrace{\left[t_i - \frac{x_i + x_{i-1}}{2}\right]}_{|\;\;| \leq \frac{1}{2}(x_i - x_{i-1})}(x_i - x_{i-1})$$



$$|\sigma-\tilde{\sigma}| \leq \sum_{i=1}^{n} \frac{1}{2}(x_i - x_{i-1})(x_i - x_{i-1}) \leq$$

$$\leq \frac{1}{2} \underbrace{\max_{i=1,\dots,n} (x_i - x_{i-1})} \underbrace{\sum_{i=1}^{n} (x_i - x_{i-1})}$$

$$\sum_{i=1}^{n} (x_i - x_{i-1}) = (\cancel{x_1} - x_0) + (\cancel{x_2} - \cancel{x_1}) + (\cancel{x_3} - \cancel{x_2}) + \dots +$$

$$+ (x_n - \cancel{x_{n-1}}) = -x_0 + x_n =$$

$$= b - a$$

$$|\sigma - \tilde{\sigma}| \leq \frac{1}{2} \boxed{\max_i (x_i - x_{i-1})} \cdot (b-a)$$

$$\downarrow n$$

$$0$$

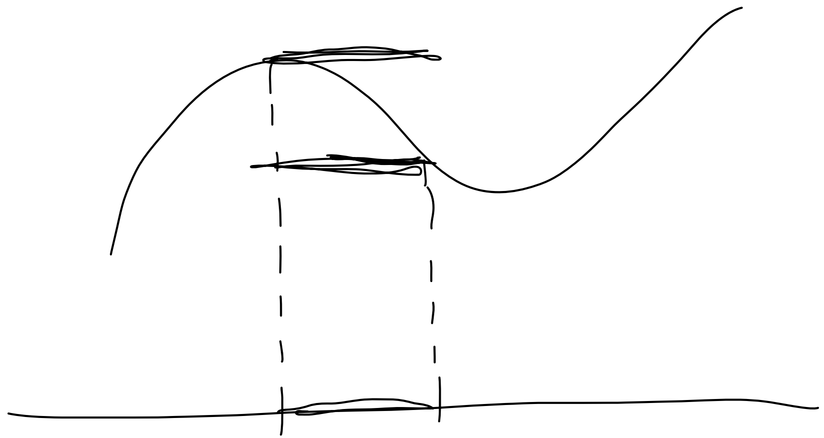
$$\sigma \xrightarrow{n \to +\infty} \tilde{\sigma} = \boxed{\frac{1}{2}(b^2 - a^2)}$$

$$\boxed{\int_a^b x \, dx = \frac{1}{2}(b^2 - a^2)}$$

Funkcje ciągłe są całkowalne

Twierdzenie

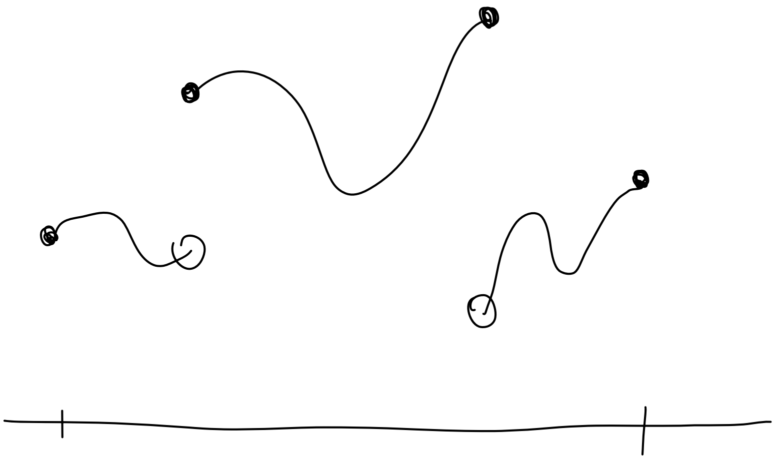
Jeżeli funkcja $f : [a, b] \to \mathbb{R}$ jest ciągła, to jest ona całkowalna.



Funkcje ciągłe są całkowalne

Twierdzenie

Jeżeli funkcja $f : [a, b] \to \mathbb{R}$ ma skończenie wiele punktów nieciągłości, to jest ona całkowalna.



Własności całki

Twierdzenie

Jeżeli funkcje f, g są całkowalne na przedziale $[a, b]$, to całkowalna jest ich suma oraz

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g.$$

$$\sigma(f+g, \mathcal{P}, \mathcal{T}) = \sigma(f, \mathcal{P}, \mathcal{T}) + \sigma(g, \mathcal{P}, \mathcal{T})$$

Własności całki

Twierdzenie

Jeżeli funkcja f jest całkowalna na przedziale $[a, b]$ oraz $c \in \mathbb{R}$, to całkowalna jest funkcja cf oraz

$$\int_a^b cf = c\int_a^b f.$$

$$\sigma(cf, \mathcal{P}, \mathcal{J}) = c\,\sigma(f, \mathcal{P}, \mathcal{J})$$

$$\int_a^b fg \qquad \int_a^b \frac{f}{g}$$