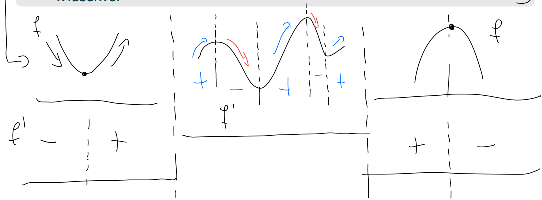
Warunek dostateczny istnienia ekstremum

Niech funkcja $f\colon (a,b) \to \mathbb{R}$ będzie **ciągła** w punkcie x_0 oraz dla pewnego $\delta > 0$ **różniczkowalna** w zbiorze $S(x_0, \delta)$. $x_0 \in (a,b)$

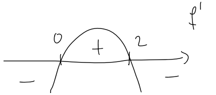
- Jeżeli $f'(x) < 0$ dla każdego $x \in (x_0 - \delta, x_0)$ oraz $f'(x) > 0$ dla każdego $x \in (x_0, x_0 + \delta)$, to f ma w punkcie x_0 minimum lokalne właściwe,
- Jeżeli $f'(x) > 0$ dla każdego $x \in (x_0 - \delta, x_0)$ oraz $f'(x) < 0$ dla każdego $x \in (x_0, x_0 + \delta)$, to f ma w punkcie x_0 maksimum lokalne właściwe.



1. $f(x) = x^2 e^{-x}$, $D = \mathbb{R}$

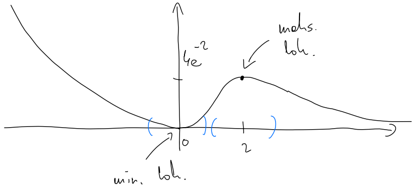
$$f'(x) = 2xe^{-x} - x^2 e^{-x} = \underbrace{e^{-x}}_{>0} \underbrace{x(2-x)}$$

$$f'(x) = 0 \iff \underbrace{x = 0 \quad \vee \quad x = 2}$$

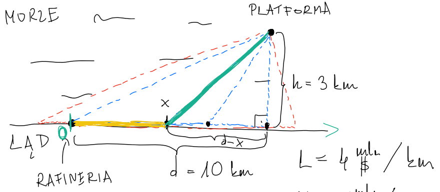


x	$(-\infty, 0)$	0	$(0, 2)$	2	$(2, +\infty)$
f'	$-$	0	$+$	0	$-$
f	↘	min. lokalne 0	↗	maks. lokalne $4e^{-2}$	↘

$f(0) = 0 \qquad f(2) = 4e^{-2}$



2.



$L = 4 \frac{mln}{\$} / km$

$M = 5 \frac{mln}{\$} / km$

funkcja kosztu

$$f : [0, d] \to \mathbb{R}$$

$$f(x) = x \cdot L + \sqrt{(d-x)^2 + h^2} \cdot M$$

$$\boxed{f'(x) = L + \frac{1}{\cancel{2}\sqrt{(d-x)^2+h^2}} \cdot \cancel{2}(d-x)(-1) \cdot M}$$

$$f'(x) = 0 \iff L - \frac{d-x}{\sqrt{(d-x)^2+h^2}} \cdot M = 0$$

$$\iff L = M \frac{d-x}{\sqrt{(d-x)^2+h^2}}$$

$$\iff L\sqrt{(d-x)^2+h^2} = M(d-x)$$

$$\iff L^2\left[(d-x)^2+h^2\right] = M^2(d-x)^2$$

$$\iff (d-x)^2(M^2-L^2) = L^2h^2$$

$$\iff (d-x)^2 = \frac{L^2h^2}{M^2-L^2}$$

$$\iff d-x = \frac{Lh}{\sqrt{M^2-L^2}}$$

$$x = d - \frac{Lh}{\sqrt{M^2-L^2}}$$

$$x = d - \frac{Lh}{\sqrt{M^2 - L^2}} \qquad d = 10 \quad L = 4. \qquad h = 3 \quad M = 5$$

$$x = 10 - \frac{4 \cdot 3}{\sqrt{5^2 - 4^2}} = 10 - \frac{4 \cdot \cancel{3}}{\cancel{3}} = 6$$

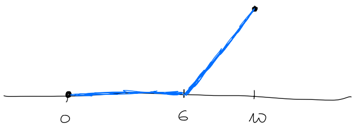
?

x	$[0,6)$	6	$(6,10]$	
f'	? $\pm$	0	$\pm$?	

$$f(0) = 0 + \sqrt{d^2 + h^2} \cdot M = \sqrt{109} \cdot 5 \approx 10.? \cdot 5 \approx 52$$

$$f(6) = 6 \cdot L + \sqrt{4^2 + 3^2} \cdot M = 24 + 25 = \boxed{49}$$

$$f(10) = 10 \cdot L + \sqrt{0^2 + 3^2} \cdot M = 40 + 15 = 55$$



Pochodne wyższych rzędów

$$f \rightsquigarrow f' \rightsquigarrow (f')' = f'' \rightsquigarrow (f'')' = f''' \rightsquigarrow (f''')' = f^{(4)} \rightsquigarrow$$

$f(x) = x^3$

$f'(x) = 3x^2$

$f''(x) = 6x$

$f'''(x) = 6$

$f^{(4)}(x) = 0$

$f^{(5)}(x) = 0$

$\vdots$

$f(x) = e^x$

$f'(x) = e^x$

$f''(x) = e^x$

$f'''(x) = e^x$

$\vdots$

$f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x$

$f^{(5)}(x) = \cos x$

$f^{(6)}(x) = -\sin x$

$f^{(7)}(x) = -\cos x$

Pochodne wyższych rzędów

Określoną indukcyjnie liczbę

$$f^{(n)}(x_0) = \begin{cases} f(x_0), & n = 0, \\ \left(f^{(n-1)}\right)'(x_0), & n \geqslant 1, \end{cases}$$

o ile istnieje, nazywamy **pochodną** ***n*****-tego rzędu** funkcji f w punkcie x_0.

$$f^{(n)} = \begin{cases} f, & n = 0, \\ \left(f^{(n-1)}\right)', & n \geqslant 1 \end{cases}$$

$$f^3 \qquad f^{(3)}$$

1. $(f+g)^{(n)} = f^{(n)} + g^{(n)}$

2. $(f\cdot g)^{(n)} = ?$

$$(f\cdot g)' = f'\cdot g + f\cdot g'$$

$$(f\cdot g)'' = (f\cdot g)^{(2)} = (f'g + fg')' =$$

$$= (f'g)' + (fg')' = f''\cdot g + f'\cdot g' + f'g' + fg'' =$$

$$= f''\cdot g + 2f'g' + f\cdot g''$$

$$(fg)''' = [f''\cdot g + 2f'g' + f\cdot g'']' =$$

$$= \underline{f'''\cdot g + f''\cdot g'} + 2(f''g' + f'g'') + (f'g'' + fg''')$$

$$= \underline{f'''g + 3f''g' + 3f'g'' + fg'''}$$

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k}$$

$$(fg)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} \cdot g^{(k)}$$

Pochodna n-tego rzędu iloczynu

Wzór Leibniza

Jeżeli funkcje f i g mają pochodne n-tego rzędu w punkcie x_0, to

$$(f \cdot g)^{(n)}(x_0) = \sum_{k=0}^{n} \boxed{\binom{n}{k}} f^{(n-k)}(x_0) \cdot g^{(k)}(x_0)$$

$$\| $$

$$\frac{n!}{(n-k)!\,k!}$$

$$f(x) = \operatorname{arctg} x \qquad f'(x) = \frac{1}{1+x^2}$$

$$f^{(n)}(x) = ? \qquad f''(x) = \frac{-1}{(1+x^2)^2} \cdot 2x = \boxed{\frac{-2x}{(1+x^2)^2}}$$

$$f'''(x) = \frac{-2(1+x^2)^2 + 2x\, 2(1+x^2)\cdot 2x}{(1+x^2)^4}$$

$$f^{(n)}(0) = ? \qquad f(0) = 0$$

$$f'(0) = 1$$

$$f''(0) = 0$$

$$f'''(0) = -2$$

$$f(x) = \operatorname{arctg} x, \quad f'(x) = \frac{1}{1+x^2}, \quad x \in \mathbb{R}$$

$$(1+x^2) f'(x) = 1 \quad \Big| \; ()^{(n)}, \quad n \geq 1$$

$$\Big(\underbrace{(1+x^2)}_{g} f'(x)\Big)^{(n)} = 0$$

$g(x) = 1+x^2$

$g'(x) = 2x$

$g''(x) = 2$

$g'''(x) = 0$

$$(g \cdot f'(x))^{(n)} = \underbrace{\binom{n}{0}}_{=1} g^{(0)} (f')^{(n)} + \underbrace{\binom{n}{1}}_{=n} g' (f')^{(n-1)} + \underbrace{\binom{n}{2}}_{=\frac{n(n-1)}{2}} g'' (f')^{(n-2)}$$

$$= (1+x^2) f^{(n+1)} + n \cdot 2x \cdot f^{(n)} + \frac{n(n-1)}{\cancel{2}} \cdot \cancel{2} f^{(n-1)}$$

$$(g \cdot f')^{(n)}(0) = f^{(n+1)}(0) + 0 + n(n-1) f^{(n-1)}(0) = 0$$

$$\boxed{f^{(n+1)}(0) = -n(n-1) f^{(n-1)}(0)}$$

$$f^{(2k)}(0) = 0 \quad \Big| \quad f'(0) = 1, \; f'''(0) = -2 \qquad f^{(2k+1)}(0) = (2k)!\,(-1)^k$$

$$f^{(5)}(0) = -4(4-1)\cdot f^{(3)}(0) = -4\cdot 3\left[-2\cdot(2-1)\cdot f^{(1)}(0)\right] =$$

$$= 4!\, f^{(1)}(0) = 4!$$

$$\boxed{f^{(2k+1)}(0) = (2k)!\,(-1)^k}$$

Wzór Taylora

$\sqrt{109} = ?$ $\sin 1 = ?$ $\log_2 3 = ?$

$w(x) = \boxed{4}x^2 \boxed{-1}x + \boxed{3}$

$w(0) = 3$ $4\cdot 2$

$w'(x) = 8x - 1$

$w'(0) = \boxed{-1}$

$w''(x) = 8 = 4\cdot 2$

$w''(0) = \boxed{4}\cdot 2$

$x^4 \to 4x^3 \to 4\cdot 3x^2 \to 4\cdot 3\cdot 2x^1$

$\to \boxed{4\cdot 3\cdot 2\cdot 1}$

$w(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$

$w(0) = a_0$

$w'(x) = n a_n x^{n-1} + \ldots + \underbrace{a_1}_{3a_3x^2 + 2a_2x}$

$w'(0) = a_1$

$w''(x) = n(n-1)a_n x^{n-2} + \ldots + \underbrace{2a_2}_{6a_3x}$

$w''(0) = 2a_2$

$w'''(x) = \boxed{n(n-1)(n-2)}a_n x^{n-3} + \ldots + 6a_3$

$w'''(0) = 6\cdot a_3$

$$\boxed{w^{(k)}(0) = k!\, a_k}$$

$$\boxed{a_k = \frac{w^{(k)}(0)}{k!}}$$

$w(0) = a_0$

$$w(x) = a_n x^n + \dots + a_1 x + a_0 =$$

$$= \frac{w^{(n)}(0)}{n!} x^n + \frac{w^{(n-1)}(0)}{(n-1)!} x^{n-1} + \dots + \frac{w^{(1)}(0)}{1!} x^1 + \frac{w^{(0)}(0)}{0!} x^0$$

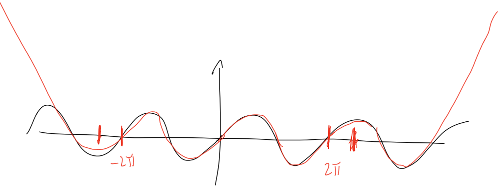
$$w(x) = \sum_{k=0}^{n} \frac{w^{(k)}(0)}{k!} x^k$$

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k}_{\text{WIELOMIAN}} + \underbrace{R(x)}_{\text{reszta}}$$

Wzór Taylora

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k + \boxed{\frac{f^{(n+1)}(c)}{(n+1)!} (x-x_0)^{n+1}}$$

c leży między x a x_0. często jest małe



$$\sqrt{3.98}\,, \qquad \sqrt{4} = 2 \qquad \sqrt{3.98} \sim 1.99$$

$$f(x) = \sqrt{x}\,, \qquad f(3.98) = \,?$$

$$x_0 = 4$$

$$f'(x) = \frac{1}{2\sqrt{x}} \qquad f'(4) = \frac{1}{4}$$

$$f''(x) = \frac{1}{2}\left(x^{-\frac{1}{2}}\right)' = -\frac{1}{4}x^{-\frac{3}{2}} \qquad f''(4) = -\frac{1}{4}\cdot\frac{1}{8} \qquad \boxed{n=2}$$

$$f'''(x) = -\frac{1}{4}\left(x^{-\frac{3}{2}}\right)' = \frac{3}{8}x^{-\frac{5}{2}}$$

$$f(x) \approx \underbrace{f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2}_{x_0=4} + \cancel{R}$$

$$\sqrt{x} \approx 2 + \underbrace{\frac{1}{4}(x-4)}_{0.005} + \frac{-\frac{1}{4}\cdot\frac{1}{8}}{2}(x-4)^2$$

$$\sqrt{3.98} \approx 2 - \frac{1}{\cancel{4}_{2}}\cdot\frac{\cancel{2}^{1}}{100} - \frac{1}{\cancel{64}_{16}}\cdot\frac{\cancel{4}^{1}}{10000} =$$

$$= 1.995 - \frac{1}{2}\cdot\frac{1}{8}\cdot\frac{1}{10000}$$

$$\frac{1}{2}\cdot\frac{0.0001}{8} = \frac{0.0000125}{2} =$$

$$= 0.00000625$$

$$= 1.9949937\boxed{5}$$

$$\sqrt{3.98} = 1.9949937\boxed{3}$$