

Problem Set 2 — Induction

1. Show that

$$1^3 + 2^3 + \dots + n^3 = \left(\frac{n(n+1)}{2} \right)^2, \quad n \in \mathbb{N}.$$

2. Find a closed form for the sum

$$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \dots + n \cdot n!$$

for all $n \in \mathbb{N}$.

3. Prove the inequality

$$1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{2^n - 1} > \frac{n}{2}, \quad n \in \mathbb{N}.$$

4. Let $(F_n)_{n \geq 1}$ be the Fibonacci sequence defined recursively by

$$F_0 = 0, \quad F_1 = 1, \quad F_{n+2} = F_{n+1} + F_n.$$

Verify that

$$F_n = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^n - \left(\frac{1 - \sqrt{5}}{2} \right)^n \right], \quad n \geq 0.$$

5. Prove that

$$F_{m+n} = F_{m+1}F_n + F_mF_{n-1}, \quad m, n \in \mathbb{N}.$$

6. Verify that

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n, \quad n \in \mathbb{N}.$$

7. Show that, for every $n \in \mathbb{N}$, the number F_{5n} is divisible by 5.

8. Define the sequence $(a_n)_{n \geq 0}$ by $a_0 = 1$, $a_1 = 3$, $a_2 = 5$, and

$$a_{n+2} = 3a_n + 2a_{n-1}, \quad n \geq 1.$$

Find a recurrence formula for a_n , $n \in \mathbb{N}$, involving no earlier term of the sequence other than a_{n-1} .

9. The sequence $(a_n)_{n \geq 0}$ is defined recursively by

$$a_1 = 1, \quad a_{m+n} + a_{m-n} = \frac{1}{2}(a_{2m} + a_{2n}), \quad m \geq n \geq 0.$$

Find an explicit formula for a_n for $n \geq 0$.

10. In a certain country, every pair of cities is connected by a one-way road. Show that there is a city in this country that can be reached from every other city either directly or by passing through at most one other city.

11. One square of size 1×1 has been removed from a chessboard of size $2^n \times 2^n$. Prove that the remaining part can be tiled with L -shaped pieces consisting of three 1×1 squares.

12. A football tournament has n teams. The tournament is played in a round-robin format, and every match ends with one of the teams winning. Show that, after the tournament, the teams can be arranged in an order such that the first team defeated the second, the second defeated the third, the third defeated the fourth, \dots , and the next-to-last defeated the last.

13. An odd number of children are standing on a snowy school playground. Each child stands in a different place, and all pairwise distances between the children are distinct. At a certain moment, each child throws a snowball at the child standing nearest to them. Prove that at least one child is not hit.

14. A pawn is placed in the lower-left corner of an $n \times n$ chessboard. In addition, a number of tiles of size 1×1 or 2×1 are placed on the board. No two tiles overlap, share a side, or share a vertex, and the square in the upper-right corner is not covered. Show that the pawn can reach this square by moving only upward or to the right and avoiding the squares occupied by tiles.

15. Show that, for every $n \in \mathbb{N}$ and every set A with n elements, all of its subsets A_i , $i = 1, \dots, 2^n$, can be arranged in an order

$$A_1, A_2, A_3, \dots, A_{2^n}$$

such that any two neighboring sets differ by exactly one element, where A_1 and A_{2^n} are also considered neighboring sets.

For $n = 3$ and the set $\{1, 2, 3\}$, one possible ordering of the subsets is:

$$\emptyset, \{1\}, \{1, 2\}, \{2\}, \{2, 3\}, \{1, 2, 3\}, \{1, 3\}, \{3\}.$$

16. Alice and Bob play the following game. There are two nonempty piles of candies on the table. Alice begins; on each move, a player eats all candies from one chosen pile and then divides the other pile into two parts, which need not be equal but must both be nonempty. Then Bob does the same: he eats one chosen pile and divides the other into two parts. The players continue in this way until one of them has no legal move. Which player has a winning strategy?