Problem Set 1 — Logic

1. Construct a formula depending on three propositional variables that takes the value 1 if and only if:

a) exactly one of the variables takes the value 1,
b) exactly two of the variables take the value 1.

2. Determine, by any method, whether the following formulas are tautologies:

a) $\{[(p \wedge q) \Rightarrow r] \wedge [(p \vee q) \Rightarrow \neg r]\} \Rightarrow (p \wedge q \wedge r)$,
b) $[(p \Rightarrow q) \wedge (r \Rightarrow q) \wedge (s \Rightarrow q)] \Rightarrow [(p \wedge r \wedge \neg s) \Rightarrow q]$,
c) $[(p \vee q) \wedge (r \vee s)] \Rightarrow \{[(p \Rightarrow q) \vee (p \Rightarrow r)] \wedge [(q \Rightarrow s) \vee (q \Rightarrow p)]\}$,
d) $[(p \Rightarrow q) \wedge (r \Rightarrow s) \wedge (t \Rightarrow u)] \Rightarrow [(p \wedge r \wedge t) \Rightarrow (q \wedge s \wedge u)]$.

3. Using known laws of propositional logic, prove that:

a) $(p \Rightarrow q) \equiv \neg(p \wedge \neg q)$,
b) $[(p \wedge q) \Rightarrow r] \equiv [p \Rightarrow (q \Rightarrow r)]$,
c) $[(p \Rightarrow r) \wedge (q \Rightarrow r)] \equiv [(p \vee q) \Rightarrow r]$.

4. Find the shortest formula equivalent to:

a) $(p \wedge q \wedge s) \vee (p \wedge \neg q \wedge \neg r) \vee (p \wedge q \wedge \neg s) \vee \neg[(p \wedge r) \Rightarrow q]$,
b) $(q \wedge r \wedge s \wedge \neg q) \vee (p \wedge \neg q \wedge \neg p) \vee (r \wedge s)$.

5. Using the connectives $\oplus$ and 1, write a formula equivalent to $p \Leftrightarrow q$.

6. Using the connectives $\oplus$, $\Rightarrow$, 1, and 0 (you do not need to use all of them), write formulas equivalent to $p \wedge q$ and $p \vee q$.

7. Express the formulas $p \oplus q$ and $p \oplus q \oplus r$ in conjunctive normal form and disjunctive normal form.

8. Express negation, conjunction, disjunction, and implication using only the NAND connective.

9. Based on the solution to Exercise 8, use the NAND connective and a single variable p to construct a formula that takes the value 1 regardless of the value of p. Construct an analogous formula that takes the value 0.

10. Express

a) the connective $\Rightarrow$ using the connectives $\vee$ and $\neg$,
b) the connective $\wedge$ using the connectives $\Rightarrow$ and $\neg$,
c) the connective $\vee$ using the connective $\Rightarrow$.

11. Justify that

a) the connective $\neg$ cannot be expressed using the connectives $\wedge$, $\vee$, $\Rightarrow$, and $\Leftrightarrow$,
b) the connective $\Rightarrow$ cannot be expressed using the connectives $\wedge$ and $\vee$,
c) the connective $\wedge$ cannot be expressed using the connectives $\vee$ and $\Rightarrow$.

12. Justify that the set of connectives $\{\Rightarrow, \neg\}$ is functionally complete.

13. Write a program that adds two positive integers without using any arithmetic operation ($+$, $-$, $\cdot$, $/$, $\ldots$).