Mathematical induction (proof technique)

$p(1), p(2), p(3), \dots, p(n), \dots$

sequence of sentences (true or false)

Induction

If

1) (BASE) $p(1)$ is TRUE

2) (INDUCTIVE STEP)

If for all $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ it follows that

$$p(n) \Rightarrow p(n+1)$$

then $p(n)$ is true for all $n \in \mathbb{N}$.

Example 1. For all $n \in \mathbb{N}$ we have

$$\underbrace{1^2 + 2^2 + 3^2 + \dots + n^2}_{L_n} = \underbrace{\frac{n(n+1)(2n+1)}{6}}_{R_n}$$

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1) $\underline{n=1}$: $\quad L_1 = R_1$?

$$L_1 = 1^2 = 1 \qquad R_1 = \frac{1(1+1)(2+1)}{6} = 1 \quad \checkmark$$

2) $\boxed{\text{Let } n \in \mathbb{N},}$

$$\underline{L_n = R_n} \implies L_{n+1} = R_{n+1} \;?$$

$$L_{n+1} = 1^2 + 2^2 + \dots + (n+1)^2 =$$

$$= \underbrace{1^2 + 2^2 + \dots + n^2}_{L_n} + (n+1)^2 =$$

BY ASSUMPTION

$$= L_n + (n+1)^2 \overset{\swarrow}{=} R_n + (n+1)^2 =$$

$$= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 =$$

$$= \frac{n(n+1)(2n+1) + 6(n+1)^2}{6} =$$

$$= \frac{(n+1)\left[n(2n+1) + 6(n+1)\right]}{6} =$$

$$= \frac{(n+1)\left[2n^2 + 7n + 6\right]}{6} = \quad \Delta = 49 - 48 = 1$$

$$= \frac{(n+1)(n+2)(2n+3)}{6} = \underline{R_{n+1}}$$

$$L_n = R_n \implies L_{n+1} = R_{n+1} \quad \checkmark$$

Example 2. (Bernoulli inequality)

For all $x \geq -1$ and all $n \in \mathbb{N}$ we have

$$(1+x)^n \geq 1 + nx.$$

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1) $\underline{n = 1}$.

$$(1+x)^1 \overset{?}{\geq} 1 + 1 \cdot x$$

$$1 + x \geq 1 + x \quad \checkmark$$

2) Let $n \in \mathbb{N}$. Assume that

$$(1+x)^n \geq 1 + nx.$$

We need to check that

$$(1+x)^{n+1} \geq 1 + (n+1)x.$$

$$\underline{(1+x)^{n+1}} = \underbrace{(1+x)^n}_{\geq 1+nx} \cdot \underbrace{(1+x)}_{\geq 0 \text{ because } x \geq -1} \geq (1+nx)(1+x) =$$

$$= 1 + x + nx + nx^2 =$$

$$= 1 + (n+1)x + \underbrace{nx^2}_{\geq 0} \geq$$

$$\geq \underline{1 + (n+1)x.}$$

<u>Strong induction</u>

If

1) (BASE) $p(1)$ is TRUE

2) (INDUCTIVE STEP)

If for all $n \in \mathbb{N} = \{1, 2, 3, \dots\}$ it follows that

$(p(1) \wedge p(2) \wedge \dots \wedge p(n)) \Rightarrow p(n+1)$

then $p(n)$ is true for all $n \in \mathbb{N}$.

Example 3. $a_1, a_2, a_3, \dots, a_n, \dots,$ $n \in \mathbb{N}$

$$\begin{cases} a_1 = 3, \quad a_2 = 5, \\ a_{n+2} = 3a_{n+1} - 2a_n, \quad n \geq 1. \end{cases} \qquad a_n = \; ?$$

Find a closed form formula for a_n.

$a_1 = 3$, $a_2 = 5$, $a_3 = 9$, $a_4 = 17$, $a_5 = 33$

2, 4, 8, 16

$$\boxed{a_n = 1 + 2^n}$$ GUESS

1) $a_1 = 3 = 1 + 2^1$, $a_2 = 5 = 1 + 2^2$ ✓

2) Assume that

$$a_k = 1 + 2^k$$

for all $k \in \{1, 2, \dots, n\}$ for some $n \in \mathbb{N}$.

We need to check that $a_{n+1} = 1 + 2^{n+1}$.

$$a_{n+1} = 3a_n - 2a_{n-1}, \quad n \geq 2$$

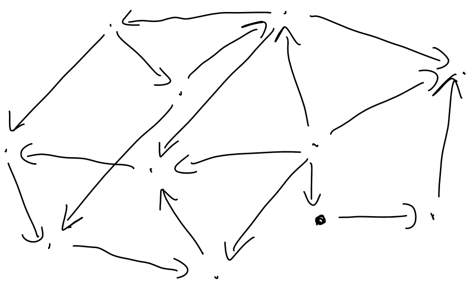
$$\underline{a_{n+1}} = 3a_n - 2a_{n-1} =$$

$$= 3(1 + 2^n) - 2(1 + 2^{n-1}) =$$

$$= 3 + 3 \cdot 2^n - 2 - 2 \cdot 2^{n-1} =$$

$$= 1 + 3 \cdot 2^n - 2^n = 1 + 2 \cdot 2^n = \boxed{1 + 2^{n+1}}$$

Example 4. In some country there are n cities. Each pair of cities is connected by a one-way road. Prove that there is a way to make a trip in this country that visit all cities.



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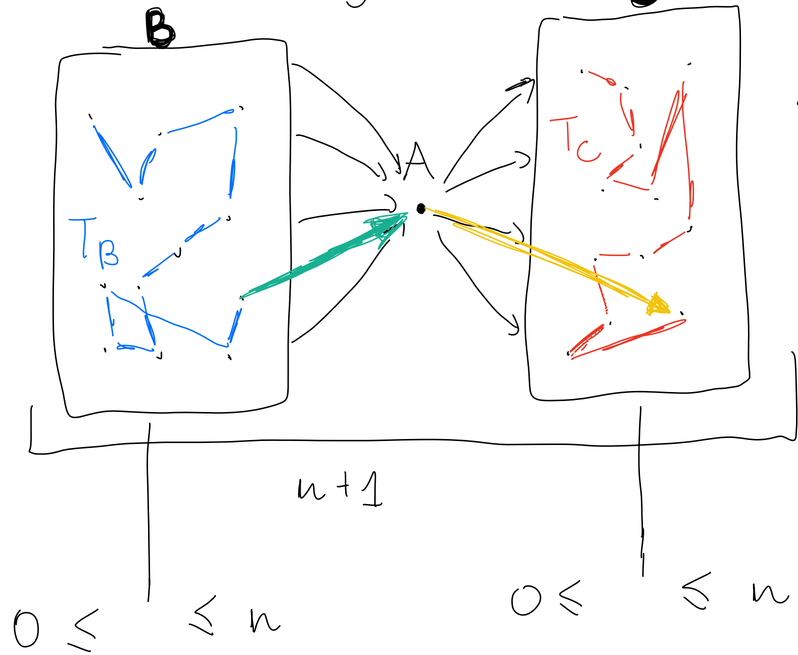
1) If there is only one city, there is nothing to prove.
2) Let $n \in \mathbb{N}$. Assume that such journey is possible for all countries with <u>at most</u> n cities.

We prove that such trip is possible for a country with $n+1$ cities.

Let A be any city.



$T_B \to A \to T_C$

Apply the assumption to B and C:
There is a trip T_B that visits all cities in B and stays in B the whole time.
The same for C.