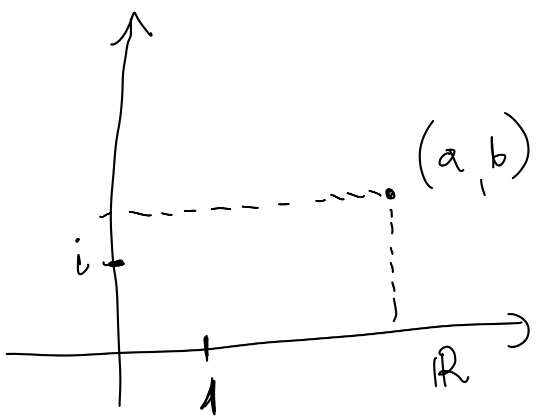
$\mathbb{C} = (\mathbb{R}^2, +, \cdot)$

$(a,b) + (c,d) = (a+c, b+d)$

$(a,b) \cdot (c,d) = (ac - bd, ad + bc)$

$\{(a,0) \in \mathbb{C} : a \in \mathbb{R}\} \cong \mathbb{R}$ $\qquad \mathbb{R} \subset \mathbb{C}$

$\mathbb{C}$



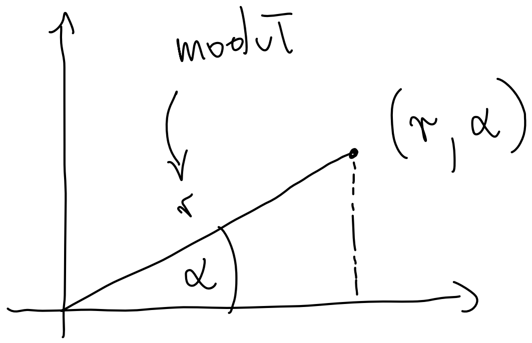

$(1,0) = 1$

$(a,0) = a$

$(0,1) = i$

$\boxed{i^2 = -1}$

$(a,b) = a(1,0) + b(0,1) = \underbrace{a + bi}_{\text{postać algebraiczna}}, \quad a, b \in \mathbb{R}$



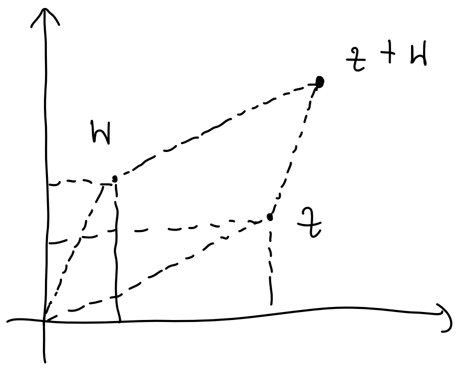

$\leftarrow$ wsp. biegunowe

$z = \underbrace{r(\cos\alpha + i\sin\alpha)}_{\text{postać trygonometryczna}}$

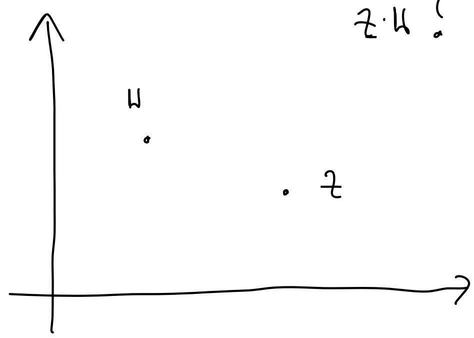
$r = |z|$

$\alpha = \arg(z)$

$(a,b) \longleftrightarrow (r,\alpha)$



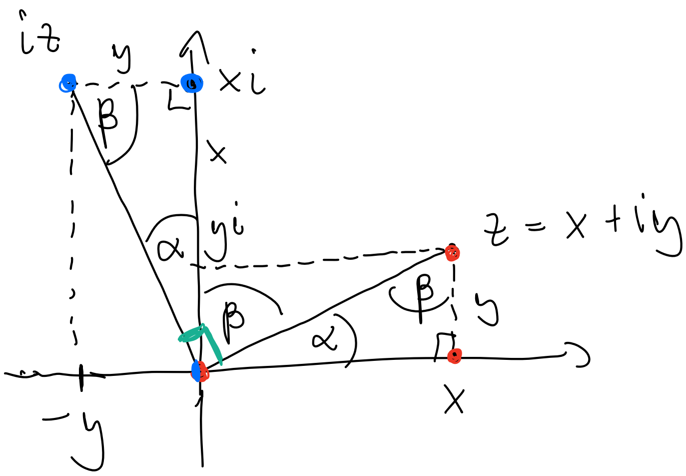

$z \cdot w$ ?




$z \in \mathbb{C}$ $\quad i \in \mathbb{C}$ $\quad iz$ ?

$z = x + iy$, $\quad x, y \in \mathbb{R}$

$iz = i(x+iy) = ix + i^2 y = -y + ix$




przystające

$\alpha + \beta = \frac{\pi}{2}$

$|z| = |iz|$

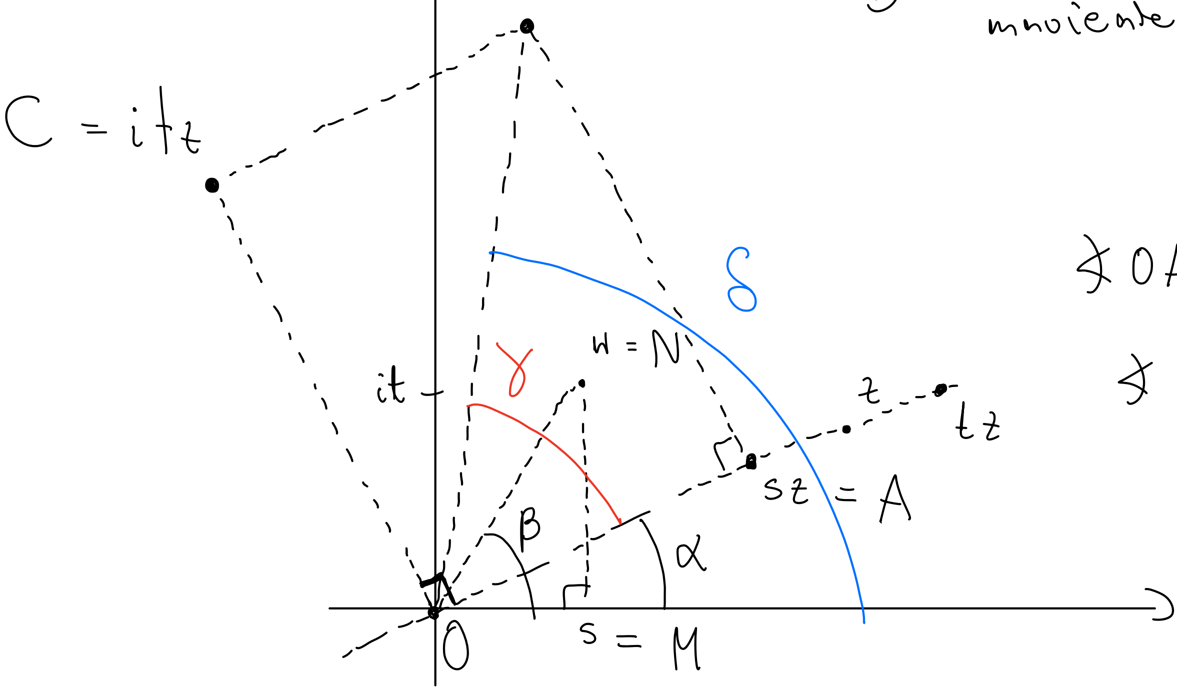
Mnożenie przez $i$ jest obrotem o kąt $90°$.

Ćw. $z$ z innej ćwiartki.

$z, w \in \mathbb{C}$ $\qquad z \cdot w\ ?$

$w = s + ti$, $\quad s, t \in \mathbb{R}$, $\quad s \neq 0$

$zw = sz + itz = B$ (gdyby $s = 0$, to mamy mnożenie przez $i$)




$\sphericalangle OAB = \sphericalangle OMN = 90^\circ$

$\sphericalangle MON = \beta$

$$z \cdot w = z(s + ti) = \underbrace{sz} + \underbrace{itz}$$

$$\operatorname{tg}\beta = \frac{|MN|}{|OM|} = \frac{|t|}{|s|} \qquad \operatorname{tg}\gamma = \frac{|AB|}{|OA|} = \frac{|OC|}{|sz|} = \frac{|itz|}{|sz|} = \frac{|tz|}{|sz|} = \frac{|t|\cdot|z|}{|s|\cdot|z|} = \frac{|t|}{|s|}$$

$$\Rightarrow \underline{\beta = \gamma} \Rightarrow \triangle OAB \sim \triangle OMN$$

podobne w skali $k$

$$\delta = \alpha + \gamma = \alpha + \beta$$

$$\Rightarrow \boxed{\arg(zw) = \arg(z) + \arg(w)}$$

---

$$k = \frac{|AB|}{|MN|} = \frac{|OC|}{|MN|} = \frac{|t|\cdot|z|}{|t|} = |z| \;\Rightarrow\; \boxed{|zw| = |OB| = k\cdot|ON| = |z|\cdot|w|}$$

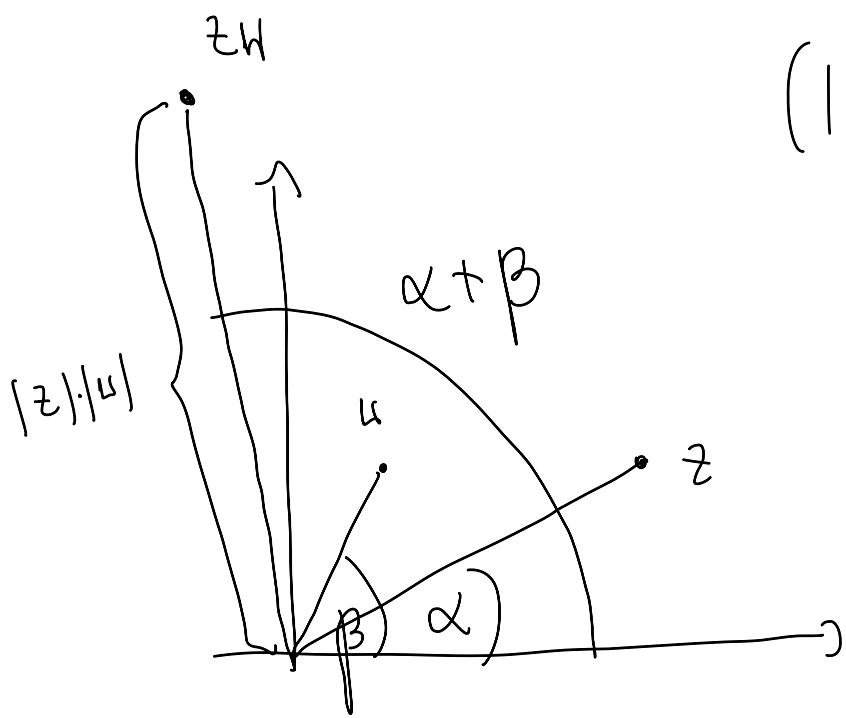
# Mnożenie w postaci trygonometrycznej

Jeżeli $z, w \in \mathbb{C}$ oraz $\quad z = (|z|, \alpha) \quad w = (|w|, \beta)$

$$z = |z|(\cos\alpha + i\sin\alpha), \qquad w = |w|(\cos\beta + i\sin\beta),$$

to

$$zw = |z||w|(\cos(\alpha+\beta) + i\sin(\alpha+\beta)).$$




$$(|z|, \alpha) \cdot (|w|, \beta) =$$

$$= (|z| \cdot |w|, \alpha + \beta)$$

vs.

$$(a, b) \cdot (c, d) =$$

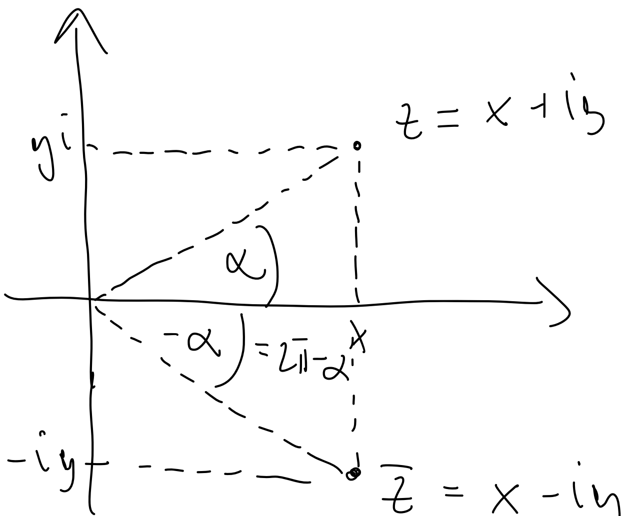
$$= (ac - bd, ad + bc)$$

$z, w \in \mathbb{C}$ , $z+w$ , $z-w = z+(-1)w$ , $zw$ ✓

$\frac{w}{z} = ?$ $\qquad z \neq 0$

$\frac{w}{z} \overset{def.}{=} w \cdot \underline{z^{-1}}$ $\qquad z^{-1}$ ?

$\underline{z^{-1} \cdot z = 1}$ ← def. odwrotności




$\bar{z} = x - iy$

$\overline{(|z|, \alpha)} = (|z|, -\alpha)$

$\overline{(x, y)} = (x, -y)$

$\overline{z+w} = \bar{z} + \bar{w}$

$\overline{zw} = \bar{z} \cdot \bar{w}$ $\qquad \overline{(x+iy)(s+ti)} = \overline{(\ldots)} = (x-iy)(s-ti)$

$\underline{z \cdot \bar{z}} = (x+iy)(x-iy) = x^2 + \cancel{ixy} - \cancel{ixy} - i^2 y^2 =$

$= x^2 + y^2 = \underline{|z|^2}$ $\qquad \boxed{|z| = \sqrt{x^2+y^2}}$

$\boxed{?} \cdot z = 1 \quad \Big| \cdot \bar{z}$

$\boxed{?}\, z\bar{z} = \bar{z}$

$\boxed{?}\, |z|^2 = \bar{z} \quad \Big| : |z|^2 , z \neq 0$

$\boxed{?} = \frac{\bar{z}}{|z|^2}$

$$\boxed{z^{-1} = \frac{1}{z} = \frac{\bar{z}}{|z|^2}}$$

- $$\frac{3+4i}{2-i} = \underbrace{\frac{3+4i}{2-i} \cdot \frac{2+i}{2+i}}_{|2+i|^2} = \frac{(3+4i)(2+i)}{|2+i|^2} =$$

$$= \frac{6+3i+8i+4\overset{=-4}{i^2}}{2^2+1^2} = \frac{2+11i}{5} =$$

$$= \frac{2}{5} + i\,\frac{11}{5}$$

$$\frac{w}{z} = \frac{w}{z} \cdot \frac{\bar{z}}{\bar{z}} \qquad \overline{2-i} = 2+i$$

---

$$\underbrace{\frac{3+4\sqrt{2}}{2-\sqrt{2}} \cdot \frac{2+\sqrt{2}}{2+\sqrt{2}}} = \dots$$

$$\boxed{\frac{1}{1+\sqrt{2}}} \cdot \frac{1-\sqrt{2}}{1-\sqrt{2}} = \frac{1-\sqrt{2}}{-1} = \boxed{\sqrt{2}-1}$$

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta) = \boxed{\cos(\alpha+\beta)} + i\,\boxed{\sin(\alpha+\beta)}$$

$$\|$$

$$\cos\alpha\cos\beta + i\cos\alpha\sin\beta + i\sin\alpha\cos\beta - \sin\alpha\sin\beta =$$

$$= \boxed{\cos\alpha\cos\beta - \sin\alpha\sin\beta} + i\left(\boxed{\cos\alpha\sin\beta + \sin\alpha\cos\beta}\right)$$

$$(\cos\alpha + i\sin\alpha)^3 = \underbrace{\underbrace{(\cos\alpha + i\sin\alpha)(\cos\alpha + i\sin\alpha)}_{\alpha+\alpha}(\cos\alpha + i\sin\alpha)}_{2\alpha+\alpha \,=\, 3\alpha}$$

$$= \boxed{\cos(3\alpha)} + i\,\boxed{\sin(3\alpha)}$$

$$\boxed{i^3 = i \cdot i^2 = -i}$$

$$\|$$

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$$

$$\cos^3\alpha + 3\cos^2\alpha\, i\sin\alpha + 3\cos\alpha\underbrace{(i\sin\alpha)^2}_{-\sin^2\alpha} + (i\sin\alpha)^3 =$$

$$= \boxed{\cos^3\alpha - 3\sin^2\alpha} + i\left(\boxed{3\cos^2\alpha\sin\alpha - \sin^3\alpha}\right)$$

# Wzór de Moivre'a

Jeżeli $z \in \mathbb{C}$ i

$$z = |z|(\cos\alpha + i\sin\alpha)$$

oraz $n \in \mathbb{N}$, to

$$z^n = |z|^n(\cos(n\alpha) + i\sin(n\alpha)).$$

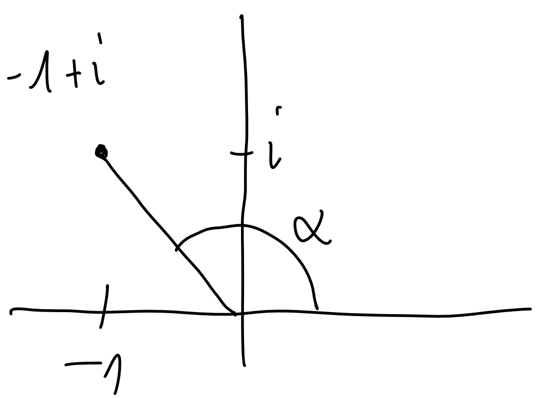
$$\frac{2025}{4} = 506 + \frac{1}{4}$$

$$3\,\frac{2025}{4} = 1518 + \frac{3}{4}$$

$$(-1+i)^{2025} = \left(\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right)^{2025} =$$

$$= \sqrt{2}^{2025}\left(\cos\left(\frac{3\pi}{4}\cdot 2025\right) + i\sin\left(\frac{3\pi}{4}\cdot 2025\right)\right) = \sqrt{2}^{2025}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$$

$$= \sqrt{2}^{2025}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -\frac{\sqrt{2}^{2026}}{2} + i\frac{\sqrt{2}^{2026}}{2} = -2^{1012} + i\,2^{1012}$$




$$|-1+i| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$$

$$\begin{cases} \cos\alpha = \dfrac{-1}{\sqrt{2}} = -\dfrac{\sqrt{2}}{2} \\ \sin\alpha = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2} \end{cases}$$

$$\alpha = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \underbrace{\frac{e^c}{(n+1)!} x^{n+1}}_{\xrightarrow{n\to\infty} 0}$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\cos x = \underline{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots}$$

$$\sin x = \underline{x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}} + \dots$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \dots =$$

$$= \underline{1} + \underline{ix} - \underline{\frac{x^2}{2!}} - i\underline{\frac{x^3}{3!}} + \underline{\frac{x^4}{4!}} + i\underline{\frac{x^5}{5!}} - \underline{\frac{x^6}{6!}} - i\underline{\frac{x^7}{7!}}$$

$$= \cos x + i \sin x$$

$$e^{ix} = \boxed{\cos x + i \sin x}$$

---

$$e^{i\pi} = \cos\pi + i\sin\pi = -1 + i \cdot 0$$

$$e^{i\pi} = -1$$

$$\boxed{e^{i\pi} + 1 = 0}$$

# Postać wykładnicza

Jeżeli $z \in \mathbb{C}$ i

$$z = |z|(\cos\alpha + i \sin\alpha),$$

to zapis

$$\|$$

$$z = |z|\mathrm{e}^{i\alpha}$$

nazywamy **postacią wykładniczą** liczby $z$.

$\sqrt{a}$ , $a \geq 0$

$$\sqrt{a} = b \iff b^2 = a \wedge b \geq 0$$

$$\sqrt{4} = 2 \cancel{\vee -2}$$

$$\sqrt[3]{a} = b \iff b^3 = a$$

$$\sqrt[n]{a} = b \iff b^n = a$$

---

$z \in \mathbb{C}$

$$\sqrt[n]{z} = w \iff w^n = z$$

# Pierwiastek zespolony

Jeżeli $z \in \mathbb{C}$ i $n \in \mathbb{N}$ to

$$\sqrt[n]{z} = \{w \in \mathbb{C} \colon w^n = z\}.$$

ten zbiór ma zawsze n el.

$$z = |z| e^{i\alpha} \qquad w = |w| e^{i\beta}$$

$$w^n = z$$

$$\left(|w| e^{i\beta}\right)^n = |z| e^{i\alpha}$$

$$|w|^n e^{in\beta} = |z| e^{i\alpha} \iff \begin{cases} |w|^n = |z| \\ n\beta = \alpha + 2k\pi, \ k \in \mathbb{Z} \end{cases}$$

$$\Leftarrow \begin{cases} |w| = \sqrt[n]{|z|} \\ \beta_k = \dfrac{\alpha + 2k\pi}{n}, \ k \in \mathbb{Z} \end{cases} \qquad k \in \{0, \ldots, n-1\}$$

$$\beta_k = \beta_{n+k} + 2\pi$$

# Pierwiastek zespolony

Jeżeli $z \in \mathbb{C}$, $z \neq 0$, to $\sqrt[n]{z}$ ma dokładnie $n$ różnych elementów. Dla

$$z = |z|(\cos\alpha + i\sin\alpha)$$

mamy

$$\sqrt[n]{z} = \{z_0, z_1, \ldots, z_{n-1}\},$$

gdzie

$$z_k = \sqrt[n]{|z|}\left(\cos\frac{\alpha + 2k\pi}{n} + i\sin\frac{\alpha + 2k\pi}{n}\right)$$

dla $k = 0, 1, \ldots, n-1$.

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C} \overset{?}{\subset} ?$$

$$\begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ x+1=0 & 3x+2=0 & x^2=2 & x^2=-1 \end{array}$$

## Zasadnicze twierdzenie algebry

Każdy wielomian stopnia $\geqslant 1$ ma pierwiastek zespolony.

# Zasadnicze twierdzenie algebry

Każdy wielomian $p$ stopnia $\geqslant 1$ ma dokładnie $n$ pierwiastków zespolonych $z_1, z_2, \ldots, z_n$ i

$$p(z) = a_n(z - z_1)(z - z_2)\ldots(z - z_n).$$