Całki niewłaściwe

- $\int_0^1 x\ln x\,dx$

$D = (0,+\infty)$

$$\lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = 0$$

$$\lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+}(-x) = 0$$

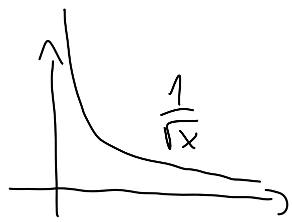
$$f(x) = \begin{cases} x\ln x, & x\in(0,1], \\ 0, & x = 0. \end{cases}$$ $\leftarrow$ f jest ciągła na $[0,1]$.

$$\int_0^1 x\ln x\,dx \overset{\text{def.}}{=} \int_0^1 f(x)\,dx$$

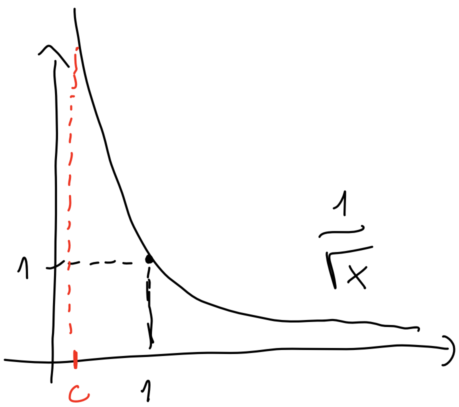
- $\int_0^1 \frac{1}{\sqrt{x}}\,dx$

$D = (0,+\infty)$

$$\lim_{x\to 0^+} \frac{1}{\sqrt{x}} = +\infty$$



$x \mapsto \frac{1}{\sqrt{x}}$ jest funkcją nieograniczoną w okolicy 0.



Dla dowolnego $c\in(0,1)$, funkcja $x\mapsto \frac{1}{\sqrt{x}}$ jest ciągła na $[c,1]$. (i ograniczona)

$\int_c^1 \frac{1}{\sqrt{x}}\,dx$ — to jest zwykła całka oznaczona

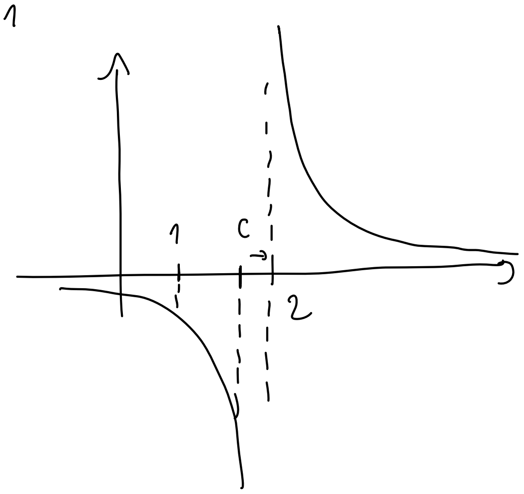
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx \overset{?}{\overset{\text{def.}}{=}} \lim_{c\to 0^+} \int_c^1 \frac{1}{\sqrt{x}}\,dx$$

$$\int_c^1 \frac{1}{\sqrt{x}}\,dx = \int_c^1 x^{-\frac{1}{2}}\,dx = \frac{x^{1/2}}{1/2}\Big|_c^1 = 2\sqrt{x}\Big|_c^1 = 2-2\sqrt{c}$$

$$\lim_{c\to 0^+}\int_c^1 \frac{1}{\sqrt{x}}\,dx = \lim_{c\to 0^+}\left(2-2\sqrt{c}\right) = 2$$

$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = 2.$$

- $$\int_1^2 \frac{1}{x-2}\,dx \overset{def.}{=} \lim_{c\to 2^-}\int_1^c \frac{1}{x-2}\,dx = \lim_{c\to 2^-} \ln|x-2|\Big|_1^c =$$



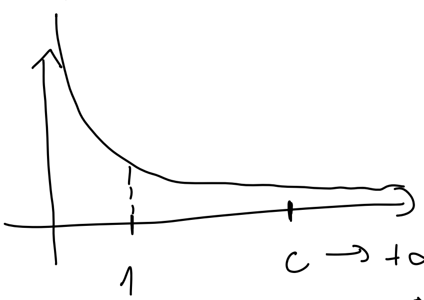
$$= \lim_{c\to 2^-}\left[\ln|c-2| - \ln|1-2|\right]$$

$$= \lim_{c\to 2^-}\left[\ln(\underbrace{2-c}_{\downarrow\, 0^+}) - \underbrace{\ln 1}_{=\,0}\right] =$$

$$= -\infty$$

- $$\int_1^{+\infty}\frac{1}{x^2}\,dx = \lim_{c\to +\infty}\int_1^c \frac{1}{x^2}\,dx = \lim_{c\to+\infty}\frac{x^{-1}}{-1}\Big|_1^c =$$

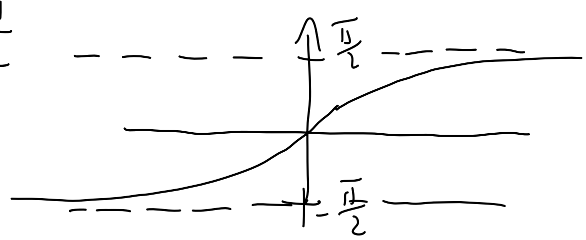
$$= \lim_{c\to+\infty}\left[\frac{-1}{c} - \frac{-1}{1}\right] = \underline{1}$$



- $$\int_{-\infty}^{+\infty}\frac{1}{1+x^2}\,dx = \int_{-\infty}^{a}\frac{1}{1+x^2}\,dx + \int_a^{+\infty}\frac{1}{1+x^2}\,dx$$

$$\int_{-\infty}^{a}\frac{1}{1+x^2}\,dx = \lim_{c\to-\infty}\int_c^a \overset{>0}{\frac{1}{1+x^2}}\,dx = \lim_{c\to-\infty} \operatorname{arctg} x\Big|_c^a =$$

$$= \lim_{c\to-\infty}\left[\operatorname{arctg} a - \underbrace{\operatorname{arctg} c}_{}\right] = \operatorname{arctg} a + \frac{\pi}{2}$$



$$\int_a^{+\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} - \operatorname{arctg} a$$

$$\Rightarrow \int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx = \left(\frac{\pi}{2} - \operatorname{arctg} a\right) + \left(\operatorname{arctg} a + \frac{\pi}{2}\right) = \pi$$

- $\displaystyle\int_0^1 \frac{1}{\sqrt{x-x^2}}\,dx$ $\qquad x - x^2 = 0 \iff x(1-x) = 0 \Rightarrow x=0 \lor x=1$

$$\overset{\shortparallel}{\int_0^{1/2}} \frac{1}{\sqrt{x-x^2}}\,dx + \int_{1/2}^1 \frac{1}{\sqrt{x-x^2}}\,dx$$

$$\int \frac{1}{\sqrt{x-x^2}}\,dx = \int \frac{dx}{\sqrt{\frac{1}{4} - \left(x-\frac{1}{2}\right)^2}} = \left| \begin{array}{l} t = 2\left(x - \frac{1}{2}\right), \quad \frac{1}{2}t = x - \frac{1}{2} \\ dt = 2dx \end{array} \right|$$

$$\overset{\shortparallel}{\frac{1}{4} - \left(x^2 - x + \frac{1}{4}\right)} = x - x^2$$

$$= \int \frac{\frac{1}{2}dt}{\sqrt{\frac{1}{4} - \frac{1}{4}t^2}} = \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + C = \arcsin(2x-1) + C$$

$$\int_0^{1/2} \frac{1}{\sqrt{x-x^2}}\,dx = \lim_{c\to 0^+} \int_c^{1/2} \frac{1}{\sqrt{x-x^2}}\,dx = \lim_{c\to 0^+} \arcsin(2x-1)\Big|_c^{1/2} =$$

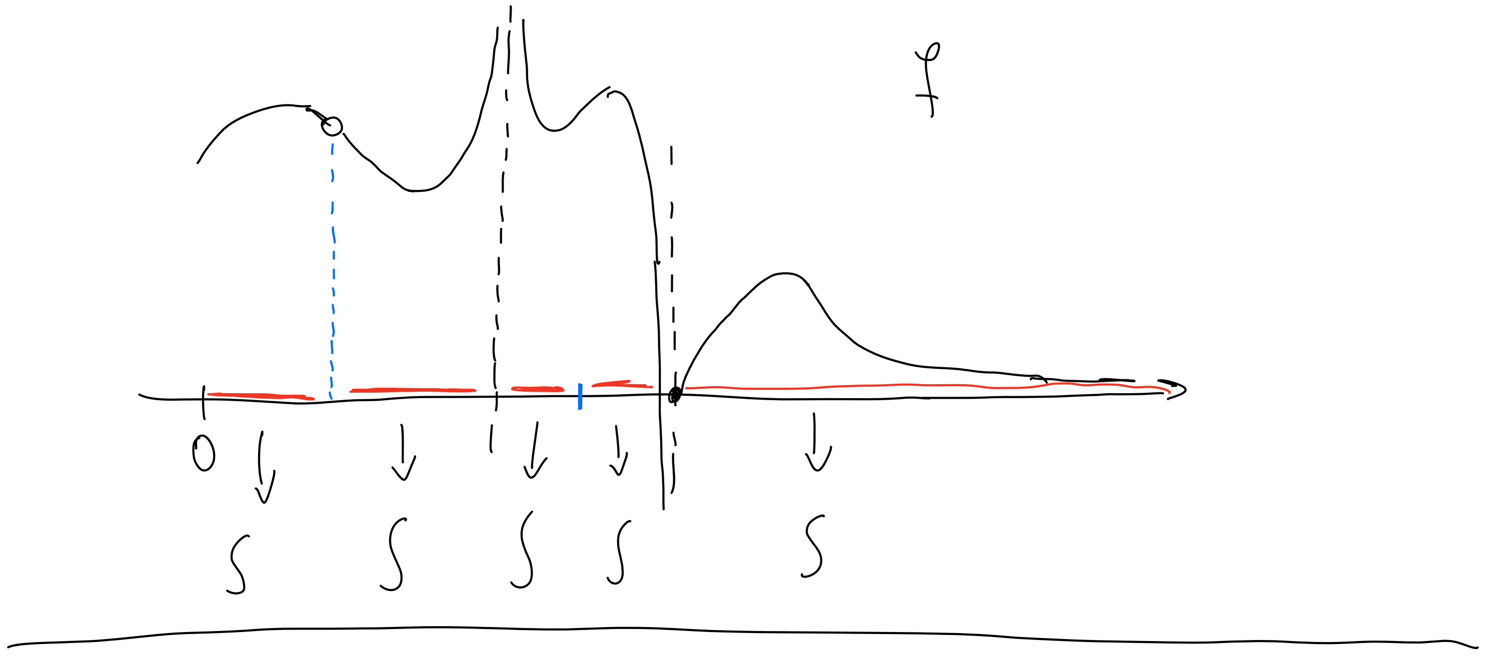
$$= \lim_{c\to 0^+}\Big[\underbrace{\arcsin(1-1)}_{0} - \arcsin(2c-1)\Big] = -\arcsin(-1) =$$

$$= \frac{\pi}{2}$$

$$\int_{1/2}^1 \frac{1}{\sqrt{x-x^2}}\,dx = \lim_{c\to 1^-}\Big[\arcsin(2c-1) - \arcsin(1-1)\Big] =$$

$$= \arcsin 1 = \frac{\pi}{2}$$

$$\int_0^1 \frac{1}{\sqrt{x-x^2}}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi$$


f
0

Liczby zespolone

1. Czy równanie

$$x+1=0$$

ma rozwiązanie?

TAK, w zbiorze $\mathbb{Z}$, $\boxed{x=-1}$ | NIE, w zbiorze $\mathbb{N}$

$\mathbb{N} \rightsquigarrow \mathbb{Z}$

$$\begin{cases} \mathbb{N}\times\mathbb{N} \quad (a,b)\sim(c,d) \Leftrightarrow a+d=b+c \\ \mathbb{Z} \overset{def.}{=} \mathbb{N}\times\mathbb{N}/\sim \end{cases}$$

2. Czy równanie

$$2x-3=0$$

ma rozwiązanie?

TAK, w zbiorze $\mathbb{Q}$ | NIE, w zbiorze $\mathbb{Z}$

$$x=\frac{3}{2}$$

$\mathbb{Z} \rightsquigarrow \mathbb{Q}$

$$\begin{cases} \mathbb{Z}\times\mathbb{Z}, \quad (a,b)\sim(c,d) \Leftrightarrow ad=bc \\ \mathbb{Q} \overset{def.}{=} \mathbb{Z}\times\mathbb{Z}/\sim \end{cases}$$

3. Czy równanie

$$x^2-2=0$$

ma rozwiązanie?

TAK, w zbiorze $\mathbb{R}$ | NIE, w zbiorze $\mathbb{Q}$

$$x=\sqrt{2} \vee x=-\sqrt{2}$$

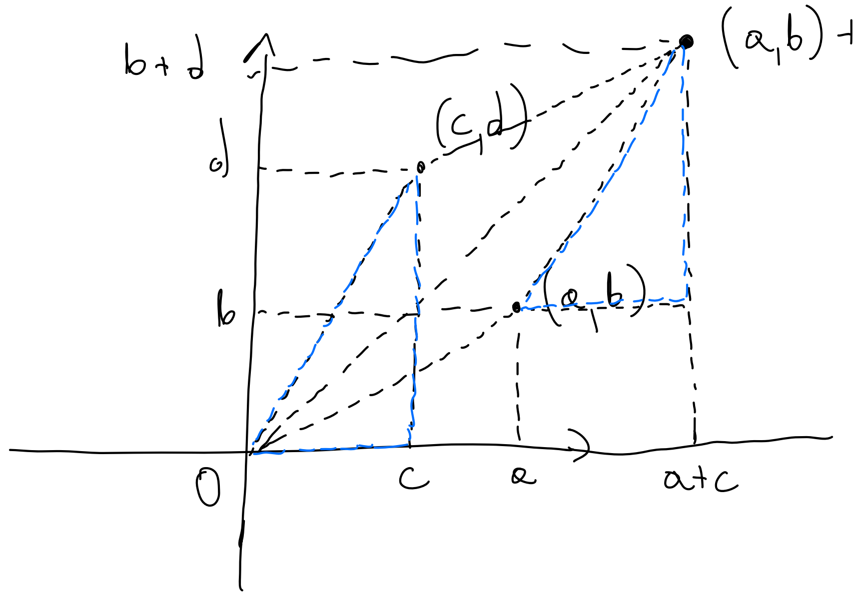
$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset ?$

4. Czy równanie
$$x^2 + 1 = 0$$
ma rozwiązanie?

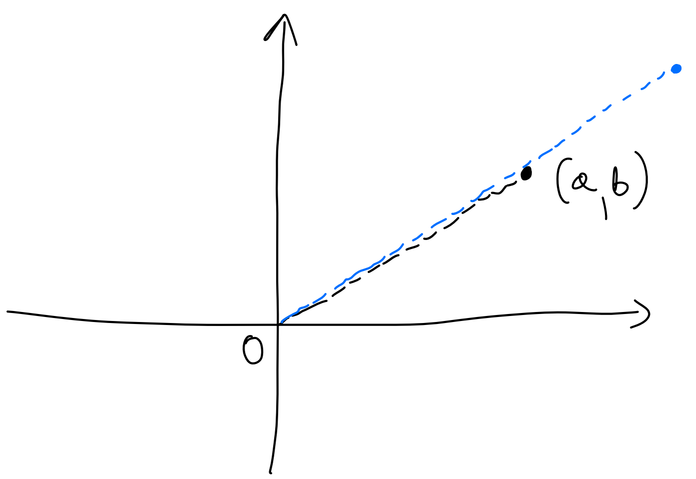
TAK, w zbiorze ? | NIE, w zbiorze $\mathbb{R}$

$\mathbb{R} \subset ?$

$\mathbb{R}^2$



$(a,b) + (c,d) = \underline{(a+c, b+d)}$



$k(a,b) = (ka, kb)$

Działania w zbiorze punktów płaszczyzny

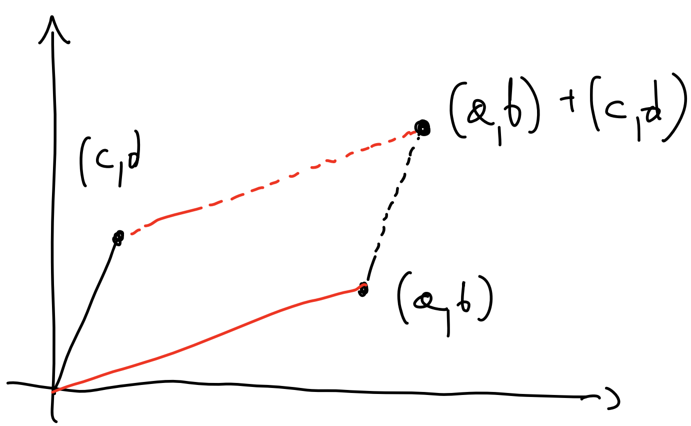
Jeżeli $(a, b) \in \mathbb{R}^2$, $(c, d) \in \mathbb{R}^2$ i $k \in \mathbb{R}$, to definiujemy

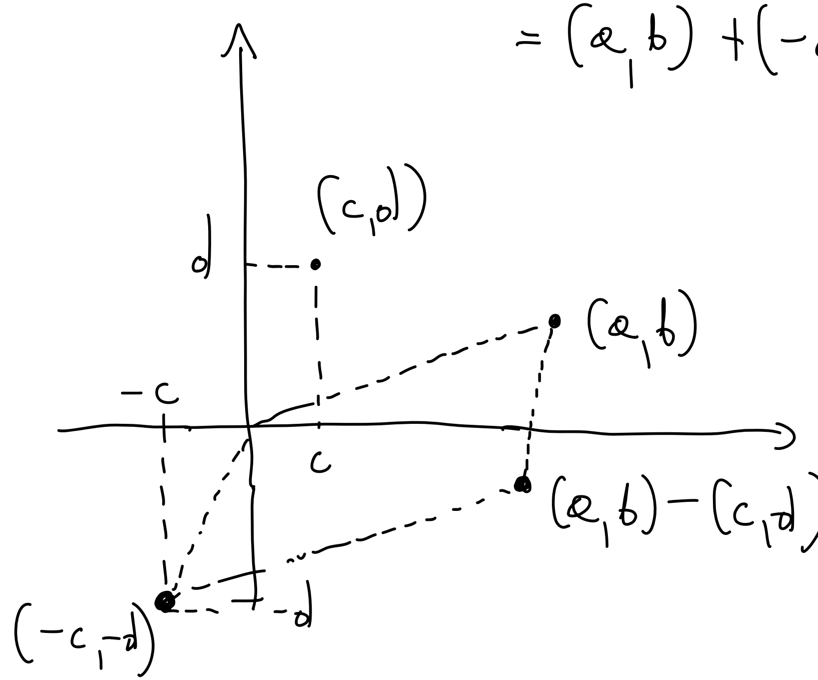
$$(a, b) + (c, d) := (a + c, b + d)$$

oraz

$$k \cdot (a, b) := (ka, kb).$$

$$(a,b) - (c,d) \overset{\text{def.}}{=} (a,b) + (-1)(c,d)$$
$$= (a,b) + (-c,-d)$$





Jak mnożyć punkty?

$$\cancel{(a,b)\cdot(c,d) = (ac, bd)} \quad ?$$

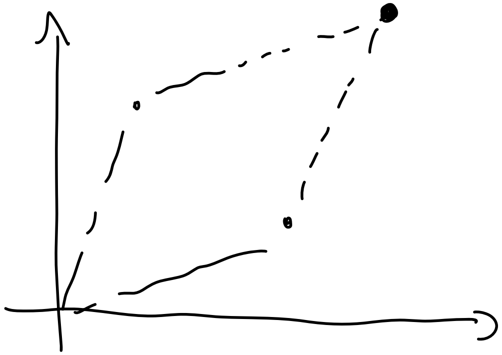
$$\cancel{\Rightarrow (2,0)\cdot(0,1) = (0,0)}$$

$$\mathbb{R} = (\mathbb{R}, +, \cdot) \rightsquigarrow (\mathbb{R}^2, +, \cdot)$$

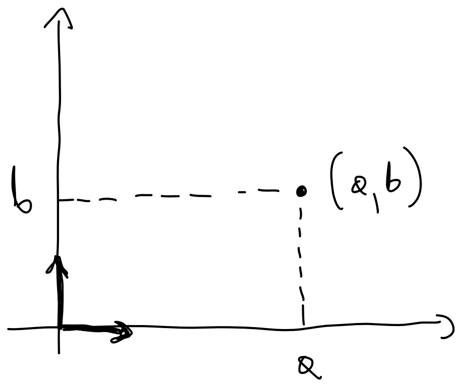
Działania w zbiorze punktów płaszczyzny

Jeżeli $(a, b) \in \mathbb{R}^2$ i $(c, d) \in \mathbb{R}^2$, to definiujemy

$$(a, b) \cdot (c, d) := \underbrace{(ac - bd, ad + bc)}_{?}.$$



?



$$(a,b) = (a,0) + (0,b) =$$
$$= a(1,0) + b(0,1)$$

$$\boxed{\begin{array}{ccc} \mathbb{R} & \longleftrightarrow & \{(a,0) : a \in \mathbb{R}\} \\ a & \longleftrightarrow & (a,0) \end{array}}$$

$$\mathbb{R} \cong \{(a,0) : a \in \mathbb{R}\}$$

$$(\mathbb{R}, +, \cdot) \overset{?}{\cong} \left(\{(a,0) : a \in \mathbb{R}\}, +, \cdot\right)$$

$$a + b \longleftrightarrow (a,0) + (b,0) = (a+b, 0)$$
$$a \cdot b \longleftrightarrow (a,0)\cdot(b,0) = (a\cdot b - 0\cdot 0, a\cdot 0 + 0\cdot b) =$$
$$= (ab, 0)$$

Twierdzenie.

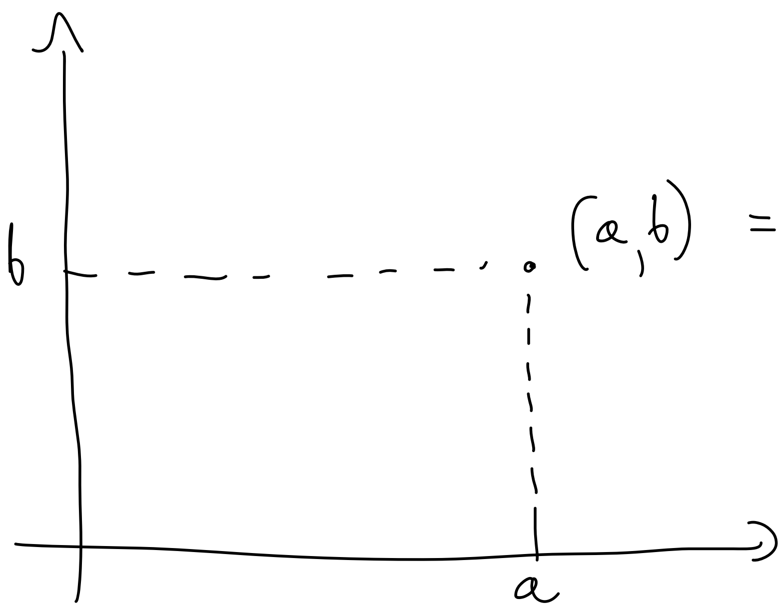
1) $+$ i $\cdot$ w $\mathbb{R}^2$ są przemienne i łączne.
2) $\cdot$ jest rozdzielne względem $+$.
   ↳ $(a,b)\cdot\big((c,d) + (e,f)\big) = (a,b)\cdot(c,d) + (a,b)\cdot(e,f)$

Dow. (ćw.!)

Zbiór liczb zespolonych

Zbiór $\mathbb{R}^2$ z działaniami dodawania i mnożenia określonymi wcześniej nazywamy **zbiorem liczb zespolonych** i oznaczamy

$$\mathbb{C} \qquad \text{lub} \qquad (\mathbb{C}, +, \cdot).$$



$$(a,b) = a\boxed{(1,0)} + b(0,1)$$

$$(1,0) \in \mathbb{C}$$

$$\updownarrow$$

$$1 \in \mathbb{R}$$

$$(a,b) = a(1,0) + b(0,1) = ?$$

$$= a \cdot 1 + b(0,1) = a + b\boxed{(0,1)}$$

Jednostka urojona

Liczbę zespoloną $(0, 1)$ nazywamy **jednostką urojoną**.

i oznaczamy przez $\mathbf{i}$.

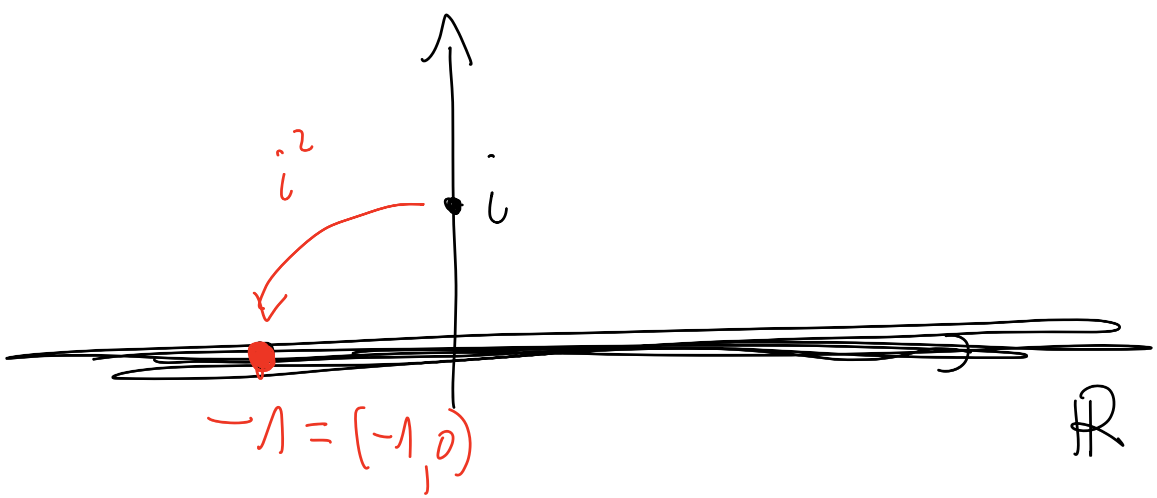
$$i \cdot i = (0,1) \cdot (0,1) = (0 \cdot 0 - 1 \cdot 1, 0 \cdot 1 + 1 \cdot 0) = (-1, 0) = -1$$

$$\boxed{i^2 = -1}$$

$$x^2 + 1 = 0$$

TAK, ma w zbiorze $\mathbb{C}$

$$x = i \vee \ldots ?$$



$$(a,b) \in \mathbb{C}$$

$$(a,b) = a\underbrace{(1,0)}_{1} + b\underbrace{(0,1)}_{i} = \boxed{a + bi}$$

$$(a,b)\cdot(c,d) = (ac - bd,\ ad + bc)$$

$$\updownarrow \qquad \updownarrow$$

$$(a+bi)(c+di) \overset{\text{tu.}}{=} a\cdot c + a\cdot di + bi\cdot c + bi\cdot di =$$

$$= ac + adi + bci + bd\underbrace{i^2}_{=-1} =$$

$$= ac + adi + bci - bd =$$

$$= ac - bd + (ad + bc)i$$

Postać algebraiczna

Jeżeli $z = (a, b) \in \mathbb{C}$, to zapis $a, b \in \mathbb{R}$

$$z = a + bi$$

nazywamy **postacią algebraiczną** liczby zespolonej z.

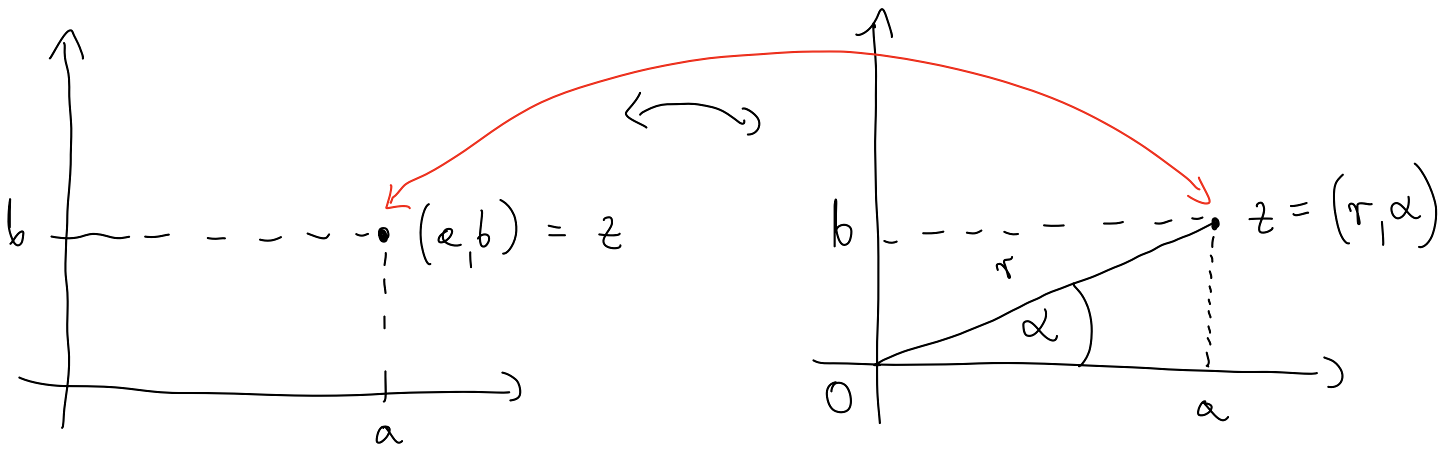
$$(2+i) + (-1+2i) = 1 + 3i$$

$$(2+i)(-1+2i) = -2 + 4i - i + 2\underbrace{i^2}_{=-1} =$$

$$= -4 + 3i$$

Układ kartezjański

Układ biegunowy



$$a + bi \longleftrightarrow \ ?$$

moduł $\longrightarrow$ $r = \sqrt{a^2 + b^2}$

$\alpha = ?$

$$\begin{cases} \cos\alpha = \dfrac{a}{r} \\ \sin\alpha = \dfrac{b}{r} \end{cases}$$

$$a + bi = r\left(\frac{a}{r} + \frac{b}{r}i\right) = r(\cos\alpha + i\sin\alpha)$$

$a + bi$

bi ← część urojona (b)

a ↑ część rzeczywista

$z = a + bi$

$a = Re(z)$ ← część rzeczywista (realis)

$b = Im(z)$ ← część urojona (imaginaris)

Postać trygonometryczna

Jeżeli $z \in \mathbb{C}$, $z \neq 0$ ma współrzędne biegunowe (r, α), to zapis

argument

$$z = r(\cos\alpha + i\sin\alpha)$$

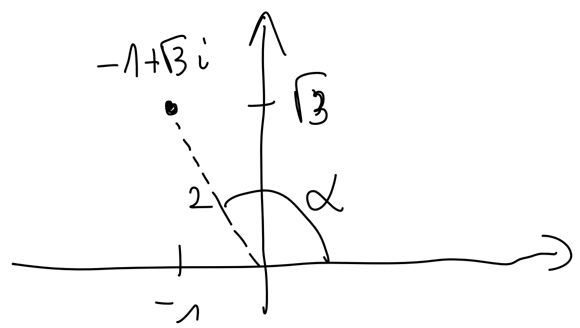
moduł

nazywamy **postacią trygonometryczną** liczby zespolonej z. Liczbę α nazywamy **argumentem** liczby z i oznaczamy $\arg z$. Jeżeli $\alpha \in [0, 2\pi)$, to liczbę tę nazywamy **argumentem głównym** i oznaczamy $\operatorname{Arg} z$.

$$a + bi \longleftrightarrow r(\cos\alpha + i\sin\alpha)$$

$$\left[\begin{array}{l} r = \sqrt{a^2 + b^2} = |z| \\ \begin{cases} \cos\alpha = \dfrac{a}{r} \\ \sin\alpha = \dfrac{b}{r} \end{cases} \end{array}\right.$$

$z = -1 + \sqrt{3}\, i$



$|z| = r = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{1+3} = 2$

$$\begin{cases} \cos\alpha = \frac{-1}{2} \\ \sin\alpha = \frac{\sqrt{3}}{2} \end{cases} \qquad \begin{cases} \cos\beta = \frac{1}{2} \\ \sin\beta = \frac{\sqrt{3}}{2} \end{cases}$$

$$\Downarrow$$

$$\alpha = \pi - \frac{\pi}{3} \quad \Longleftarrow \quad \beta = \frac{\pi}{3}$$

$$\boxed{\alpha = \frac{2\pi}{3}}$$

$$z = -1 + \sqrt{3}\, i = 2\left(\cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3}\right)$$

$$(-1, \sqrt{3}) \longleftrightarrow \left(2, \frac{2\pi}{3}\right)$$

$z, w \in \mathbb{C}$ $\quad$ $z \cdot w$ geometrycznie ?

Mnożenie w postaci trygonometrycznej

Jeżeli $z, w \in \mathbb{C}$ oraz $z = (|z|, \alpha)$ $w = (|w|, \beta)$

$$z = |z|(\cos\alpha + i\sin\alpha), \qquad w = |w|(\cos\beta + i\sin\beta),$$

to

$$zw = |z||w|(\cos(\alpha+\beta) + i\sin(\alpha+\beta)).$$