- $$\int x\sqrt{1-x^2}\,dx = \left| \begin{array}{l} t = 1-x^2 \\ dt = -2x\,dx \end{array} \right| = \int \sqrt{t}\left(-\frac{1}{2}\right)dt =$$

$$= -\frac{1}{2}\int t^{1/2}\,dt = -\frac{1}{2}\,\frac{t^{3/2}}{3/2} + C = -\frac{1}{3}t^{3/2} + C =$$

$$= \boxed{-\frac{1}{3}\left(1-x^2\right)^{3/2} + C}$$

$$\int_0^1 x\sqrt{1-x^2}\,dx \overset{N-L}{=} -\frac{1}{3}\left(1-x^2\right)^{3/2}\Big|_0^1 = -\frac{1}{3}\left(1-1\right)^{3/2} - \left(-\frac{1}{3}\left(1-0\right)^{3/2}\right) =$$

$$= \boxed{\frac{1}{3}}$$

$$\int f'g = fg - \int fg'$$

$$\int_a^b f'g = \ldots$$

Całkowanie przez części

Jeżeli funkcje f i g mają ciągłe pochodne na przedziale $[a, b]$, to

$$\int_a^b f'g = fg\Big|_a^b \cancel{+} - \int_a^b fg'$$

$$(fg)' = f'g + fg' \implies \int (fg)' = fg + C$$

$$\int_a^b (fg)' = \int_a^b f'g + \int_a^b fg'$$

$\parallel$ N-L

$$fg\Big|_a^b \implies \int_a^b f'g = fg\Big|_a^b - \int_a^b fg'$$

$$\bullet \quad \int_1^2 x^3 \ln x\,dx = \int_1^2 \left(\frac{x^4}{4}\right)' \ln x\,dx =$$

$$= \frac{x^4}{4}\ln x\Big|_1^2 - \int_1^2 \frac{x^4}{4}\cdot\frac{1}{x}\,dx =$$

$$= \frac{x^4}{4}\ln x\Big|_1^2 - \frac{1}{4}\int_1^2 x^3\,dx = \frac{x^4}{4}\ln x\Big|_1^2 - \frac{1}{16}x^4\Big|_1^2 =$$

$$= \frac{2^4}{4}\ln 2 - \frac{1^4}{4}\underset{0}{\underset{\shortparallel}{\ln 1}} - \frac{1}{16}\left[2^4 - 1^4\right] =$$

$$= 4\ln 2 - \frac{1}{16}15 = \ln 16 - \frac{15}{16}$$

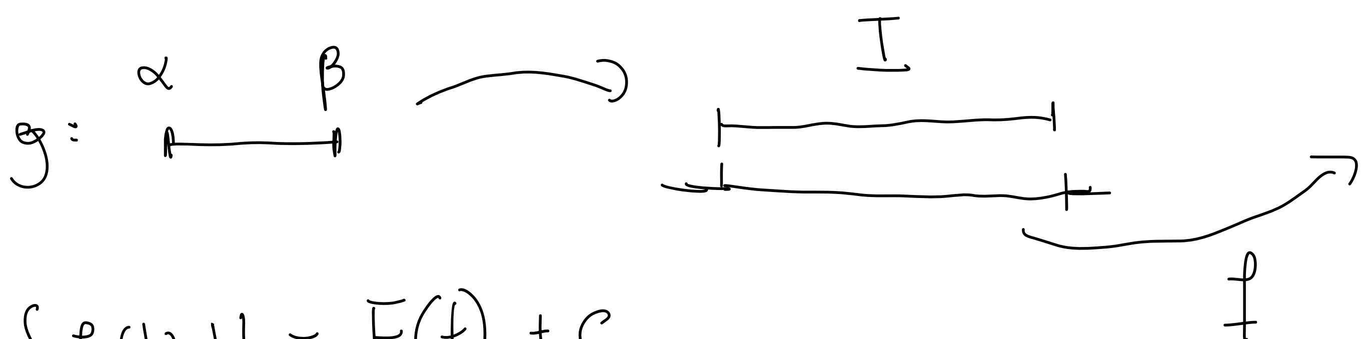
Całkowanie przez podstawienie

Jeżeli

- funkcja g jest określona i ciągła w przedziale $[\alpha, \beta]$,
- funkcja g ma ciągłą pochodną w przedziale $[\alpha, \beta]$,
- funkcja f jest ciągła na zbiorze wartości funkcji g,

to dla $a = g(\alpha)$ i $b = g(\beta)$ mamy

$$\int_{\alpha}^{\beta} f(g(x))g'(x)\,dx = \int_{a}^{b} f(t)dt.$$



$$\int f(t)\,dt = F(t) + C$$

$$\int_{\alpha}^{\beta} f(g(x))g'(x)\,dx = F(g(x))\Big|_{\alpha}^{\beta} = F(b) - F(a) = \int_{a}^{b} f(t)dt.$$

- $$\int_2^4 \frac{1}{x\ln^2 x}\,dx = \left| \begin{array}{l} t = \ln x \\ dt = \frac{1}{x}dx \\ \begin{array}{c|c|c} x & 2 & 4 \\ \hline t & \ln 2 & \ln 4 \end{array} \end{array} \right| = \int_{\ln 2}^{\ln 4} \frac{1}{t^2}\,dt =$$

$$= \int_{\ln 2}^{\ln 4} t^{-2}\,dt = \frac{t^{-1}}{-1}\Bigg|_{\ln 2}^{\ln 4} = -\frac{1}{t}\Bigg|_{\ln 2}^{\ln 4} =$$

$$I \qquad = -\frac{1}{\ln 4} - \left(-\frac{1}{\ln 2}\right) = \frac{1}{\ln 2} - \frac{1}{\ln 4}$$

$$\|$$

- $$\boxed{\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx} = \left| \begin{array}{l} x = \pi - t \\ dx = -dt \\ \begin{array}{c|c|c} x & 0 & \pi \\ \hline t & \pi & 0 \end{array} \end{array} \right| = -\int_{\pi}^{0} \frac{(\pi-t)\sin(\pi-t)}{1+\cos^2(\pi-t)}\,dt =$$

$$t = \pi - x$$

$$= \int_0^{\pi} \frac{(\pi-t)\sin(\pi-t)}{1+\cos^2(\pi-t)}\,dt = \int_0^{\pi} \frac{(\pi-t)\sin t}{1+(-\cos t)^2}\,dt =$$

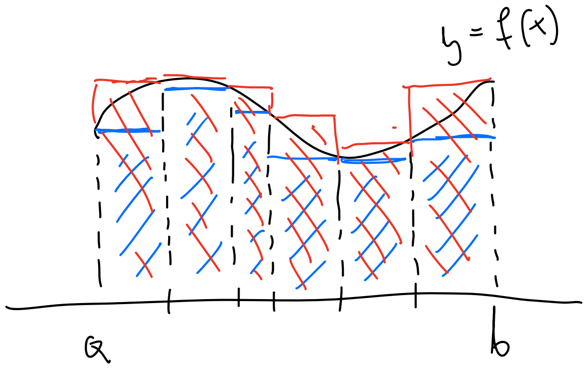
$$= \pi\int_0^{\pi} \frac{\sin t}{1+\cos^2 t}\,dt - \boxed{\int_0^{\pi} \frac{t\sin t}{1+\cos^2 t}\,dt}$$

$$\boxed{I = \frac{\pi}{2}\int_0^{\pi} \frac{\sin t}{1+\cos^2 t}\,dt} = \frac{\pi}{2}\cdot\frac{\pi}{2} = \boxed{\frac{\pi^2}{4}}$$

$$\int_0^{\pi} \frac{\sin t}{1+\cos^2 t}\,dt = \left| \begin{array}{l} \cos t = u \\ -\sin t\,dt = du \\ \begin{array}{c|c|c} t & 0 & \pi \\ \hline u & 1 & -1 \end{array} \end{array} \right| = \int_1^{-1} \frac{-1}{1+u^2}\,du =$$

$$= \int_{-1}^{1} \frac{du}{1+u^2} = \operatorname{arctg} u\Big|_{-1}^{1} =$$

$$= \operatorname{arctg} 1 - \operatorname{arctg}(-1) = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \frac{\pi}{2}$$

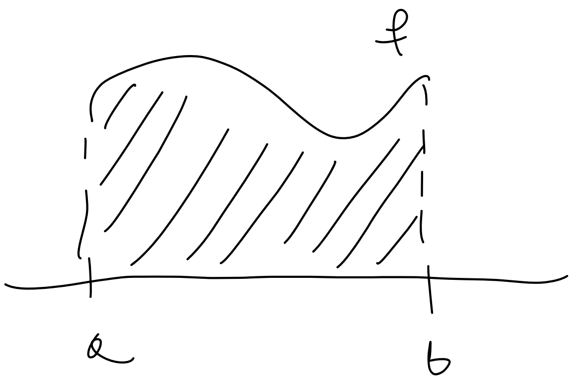


$$\underline{P} \leq P \leq \overline{P}$$

$$\underline{P} \xrightarrow[\delta \to 0]{n \to +\infty} \int_a^b f(x)\,dx \qquad \overline{P} \xrightarrow[\delta \to 0]{n \to +\infty} \int_a^b f(x)\,dx$$

$P =$ „pole pod wykresem funkcji f na przedziale $[a,b]$"

$$\underset{=}{\overset{\text{def.}}{}} \int_a^b f(x)\,dx$$



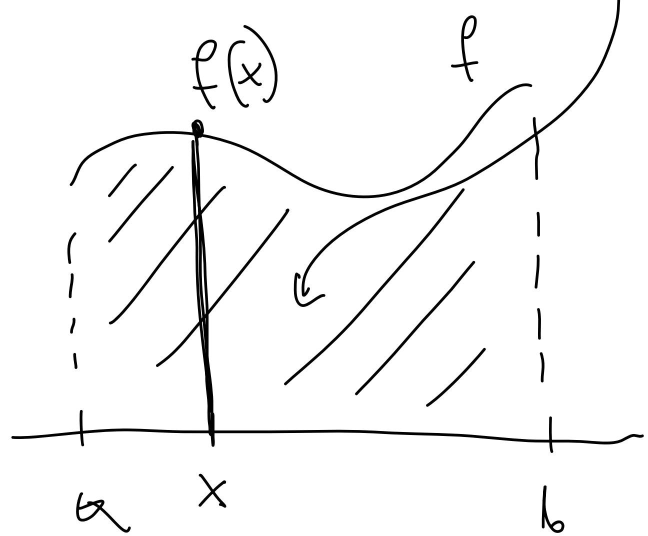
Całka jako pole

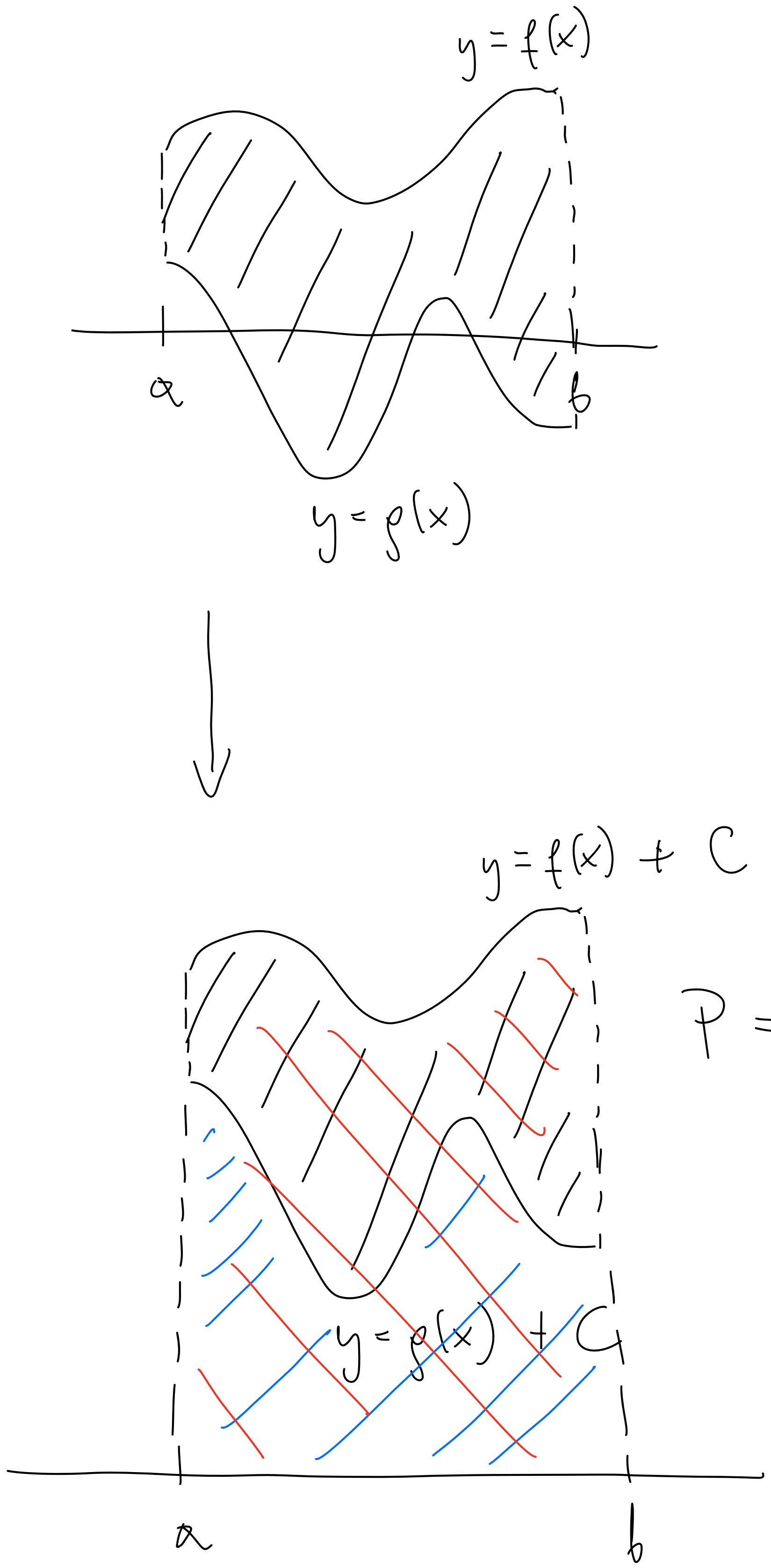
Jeżeli funkcja f jest ciągła i nieujemna na przedziale $[a, b]$, to polem obszaru

$$\{(x, y) \in \mathbb{R}^2 : x \in [a, b],\ 0 \leqslant y \leqslant f(x)\}$$

nazywamy całkę

$$\int_a^b f(x)\, dx.$$





$P = P - P$

Całka jako pole

Jeżeli funkcje f i g są ciągłe na przedziale $[a, b]$ oraz

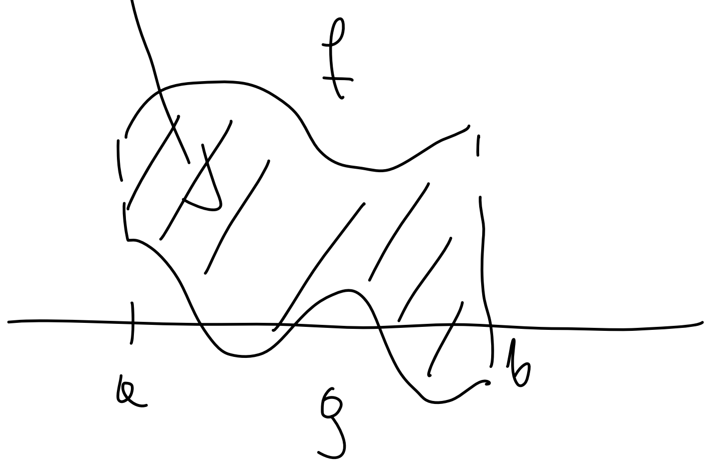
$$g(x) \leqslant f(x), \qquad x \in [a, b],$$

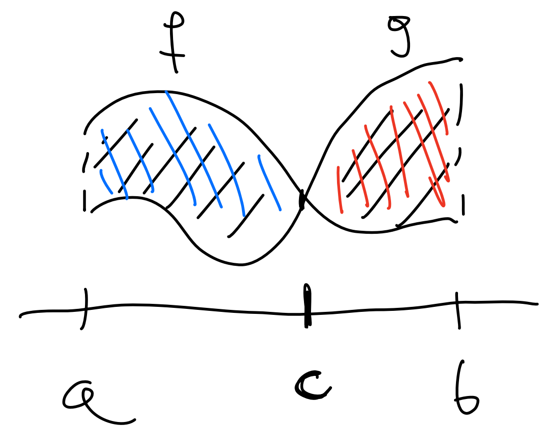
to polem obszaru

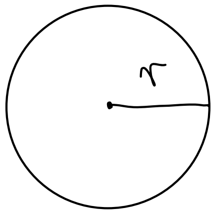
$$\{(x, y) \in \mathbb{R}^2 : x \in [a, b],\ g(x) \leqslant y \leqslant f(x)\}$$

nazywamy całkę

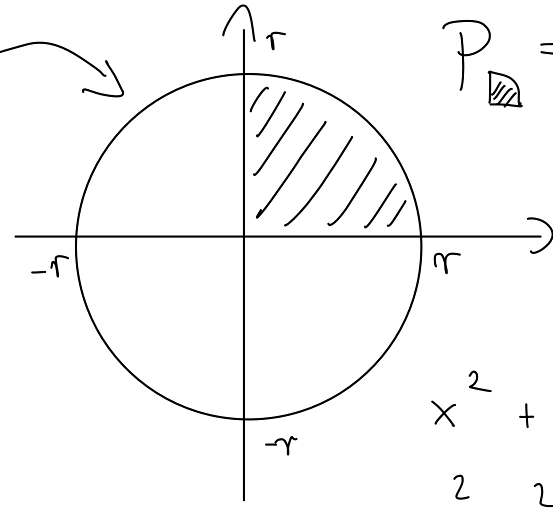
$$\int_a^b (f(x) - g(x))\, dx.$$



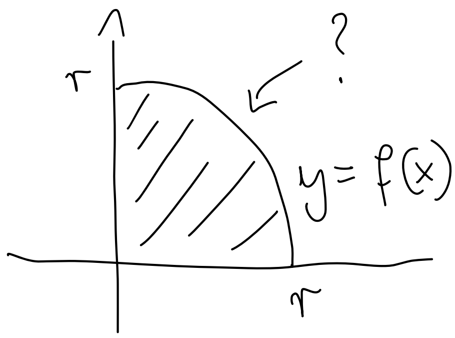




$$P = \pi r^2$$



$$P_{\text{quarter}} = ?$$



$$y = f(x) = \sqrt{r^2 - x^2}$$

$$x^2 + y^2 = r^2$$

$$y^2 = r^2 - x^2$$

$$|y| = \sqrt{r^2 - x^2}$$

równanie górnego półokręgu $\longleftarrow$ $y = \sqrt{r^2 - x^2}$

$$\underline{P_{\text{quarter}}} = \int_0^r \sqrt{r^2 - x^2}\,dx = \left| \begin{array}{l} x = r\sin t \\ dx = r\cos t\,dt \end{array} \quad \begin{array}{c|c|c} x & 0 & r \\ \hline t & 0 & \pi/2 \end{array} \right|$$

$$= \int_0^{\pi/2} \sqrt{r^2 - r^2\sin^2 t}\; r\cos t\,dt = r\int_0^{\pi/2} \sqrt{r^2(1 - \sin^2 t)}\,\cos t\,dt =$$

$$= r^2 \int_0^{\pi/2} \sqrt{\cos^2 t}\,\cos t\,dt \underset{=}{\overset{t \in [0, \pi/2]}{=}} r^2 \int_0^{\pi/2} \cos t \cdot \cos t\,dt$$

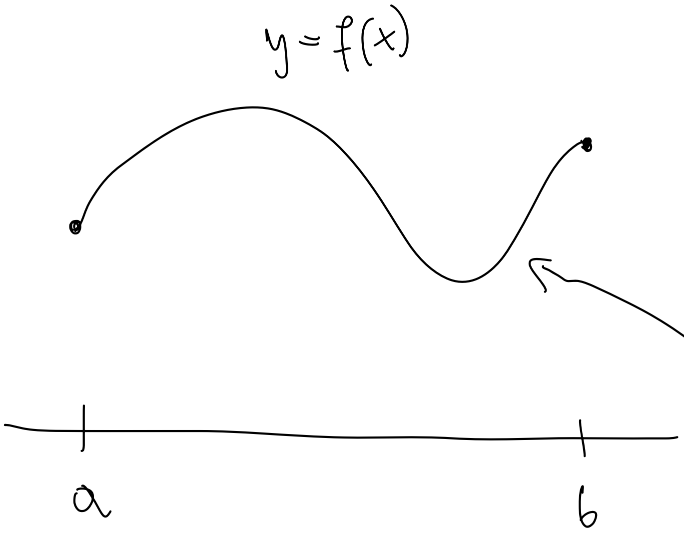
$$= r^2 \int_0^{\pi/2} \cos^2 t\,dt = r^2 \int_0^{\pi/2} \frac{1 + \cos 2t}{2}\,dt =$$

$$= \frac{r^2}{2}\int_0^{\pi/2} dt + \frac{r^2}{2}\int_0^{\pi/2} \cos 2t\,dt = \frac{r^2}{2}\cdot\frac{\pi}{2} + \frac{r^2}{2}\cdot\frac{1}{2}\sin(2t)\Big|_0^{\pi/2}$$

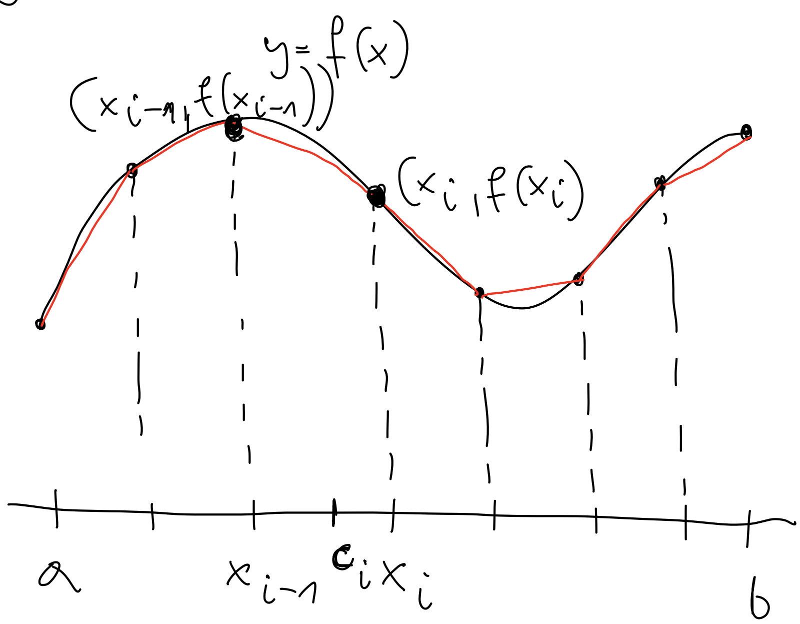
$$= \frac{\pi r^2}{4} + \frac{r^2}{4}\left(\sin\pi - \sin 0\right) = \frac{\pi r^2}{4} + \frac{r^2}{4}\cdot 0 =$$

$$= \underline{\frac{\pi r^2}{4}}$$

$$\Longrightarrow P_{\text{circle}} = 4P_{\text{quarter}} = 4\cdot\frac{\pi r^2}{4} = \boxed{\pi r^2}$$



Jaka jest długość krzywej ?



$$\text{dł}(\sim) \approx \text{dł}(\sim)$$

$$\text{dł}(\sim) = \sum_{i=1}^{n} \text{dł}\Big(\underset{x_{i-1}\quad x_i}{\diagdown}\Big) = \sum_{i=1}^{n} \sqrt{(x_i - x_{i-1})^2 + \big(f(x_i) - f(x_{i-1})\big)^2}$$

$$= \sum_{i=1}^{n} (x_i - x_{i-1}) \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2} =$$

tw. Lagrange'a na $[x_{i-1}, x_i]$

$$= \sum_{i=1}^{n} \Delta x_i \underbrace{\sqrt{1 + \big(f'(c_i)\big)^2}}_{c_i \in (x_{i-1}, x_i)} \xrightarrow[\delta \to 0]{n \to +\infty} \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

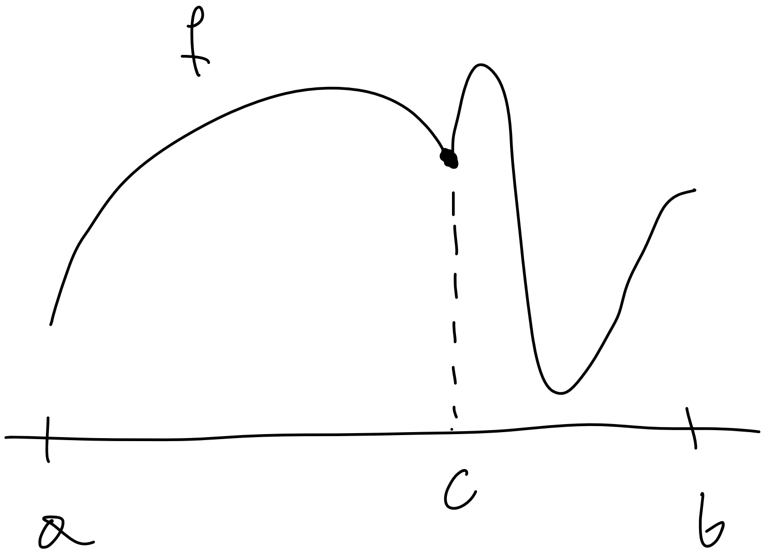
Długość krzywej

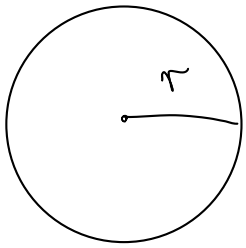
Jeżeli funkcja f ma ciągłą pochodną na przedziale $[a, b]$, to długością krzywej

$$\{(x, y) \in \mathbb{R}^2 \colon x \in [a, b],\ y = f(x)\}$$

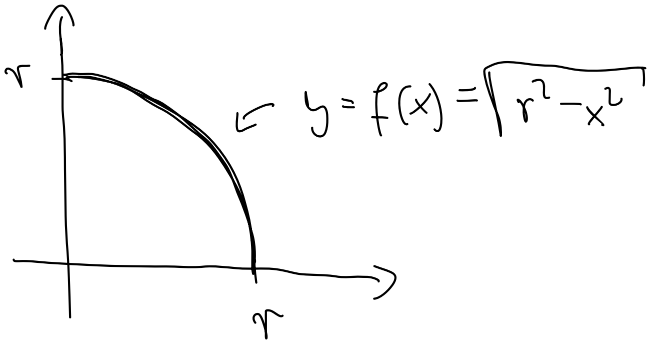
nazywamy całkę

$$\int_a^b \sqrt{1 + (f'(x))^2}\, dx.$$





$L = 2\pi r$



$$f(x) = \sqrt{r^2 - x^2} \qquad f'(x) = \frac{1}{2\sqrt{r^2-x^2}} \cdot (-2x) = \frac{-x}{\sqrt{r^2-x^2}}$$

$$L_{\frown} = \int_0^r \sqrt{1 + \left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\, dx = \int_0^r \sqrt{\frac{r^2-x^2+x^2}{r^2-x^2}}\, dx =$$

$$= \int_0^r \sqrt{\frac{r^2}{r^2-x^2}}\, dx = r\int_0^r \frac{dx}{\sqrt{r^2-x^2}} =$$

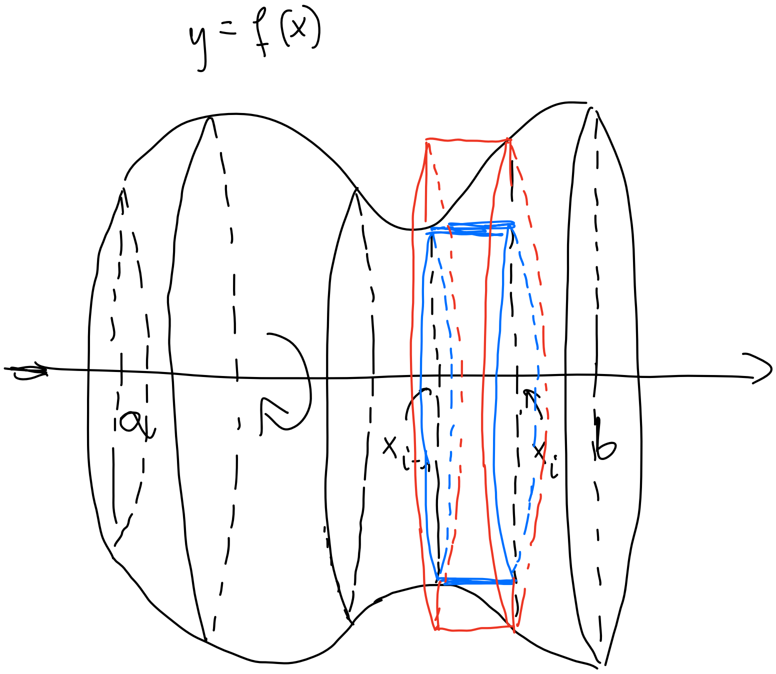
$$= \left| \begin{array}{l} x = rt \\ dx = r\,dt \\ \begin{array}{c|c|c} x & 0 & r \\ \hline t & 0 & 1 \end{array} \end{array} \right| = r\int_0^1 \frac{r\,dt}{\sqrt{r^2 - r^2t^2}} = r^2 \int_0^1 \frac{dt}{r\sqrt{1-t^2}} =$$

$$= r\int_0^1 \frac{dt}{\sqrt{1-t^2}} = r \arcsin t \Big|_0^1 =$$

$$= r\left[\arcsin 1 - \arcsin 0\right] = r\left[\frac{\pi}{2} - 0\right] =$$

$$= \frac{\pi r}{2}$$

$$\Rightarrow L_{\bigcirc} = 4L_{\frown} = 4 \cdot \frac{\pi r}{2} = \boxed{2\pi r}$$



$$V \le V \le V$$

$$V = \sum_{i=1}^{n} V_{\square} = \sum_{i=1}^{n} \pi \cdot \Big(\underbrace{f(c_i)}_{\text{poloměr podstavy válce}}\Big)^2 \cdot \overbrace{(x_i - x_{i-1})}^{\Delta x_i}$$

$$\xrightarrow[\delta \to 0]{n \to +\infty} \int_a^b \pi \big(f(x)\big)^2 \, dx = \pi \int_a^b f^2(x)\,dx$$

$$V = \sum_{i=1}^{n} V_{\square} \xrightarrow[\delta \to 0]{n \to +\infty} \pi \int_a^b f^2(x)\,dx$$

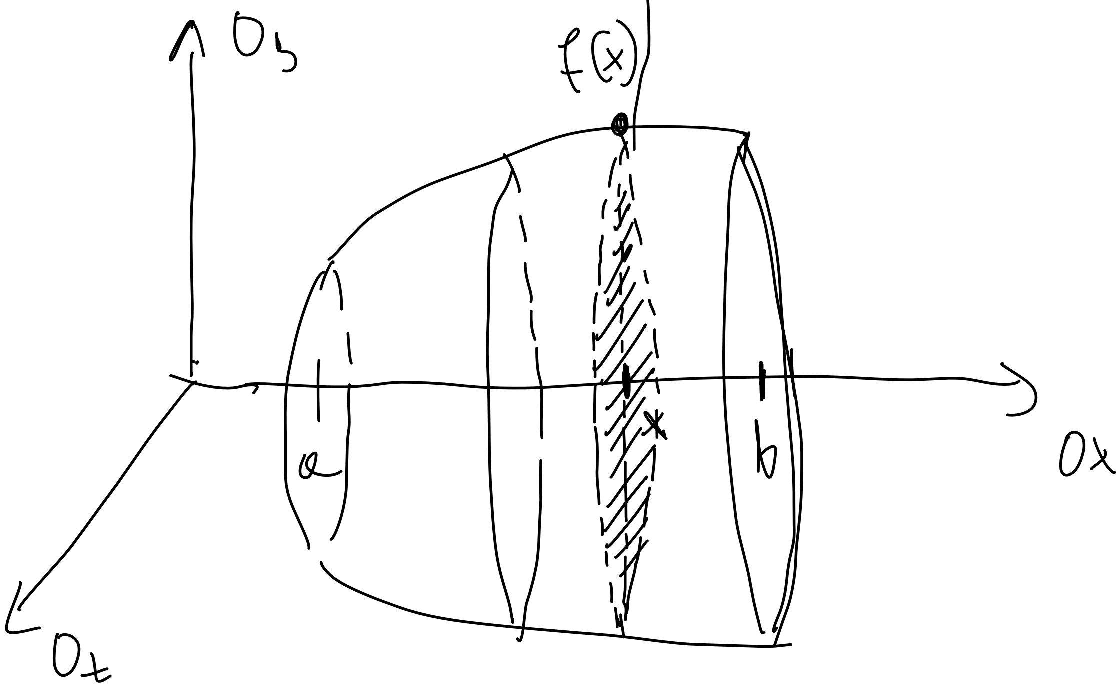
Objętość bryły obrotowej

Jeżeli funkcja f jest ciągła i nieujemna na przedziale $[a, b]$, to objętością bryły obrotowej

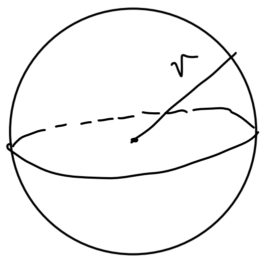
$$\{(x, y, z) \in \mathbb{R}^3 \colon x \in [a, b],\ y^2 + z^2 \leqslant f^2(x)\}$$

nazywamy liczbę

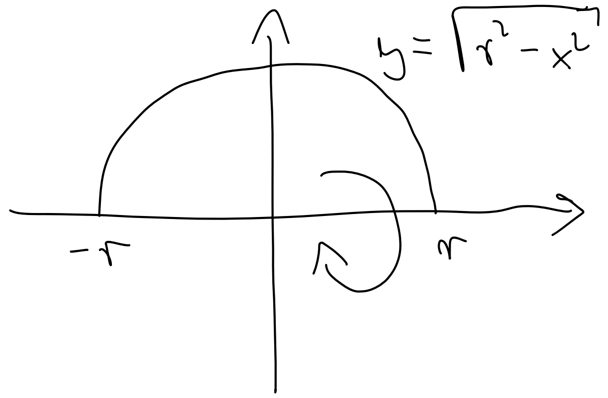
$$\pi \int_a^b f^2(x)\, dx.$$



koło w pł. Oyz
o środku (0,0)
i promieniu f(x)



$$V = \frac{4}{3}\pi r^3$$



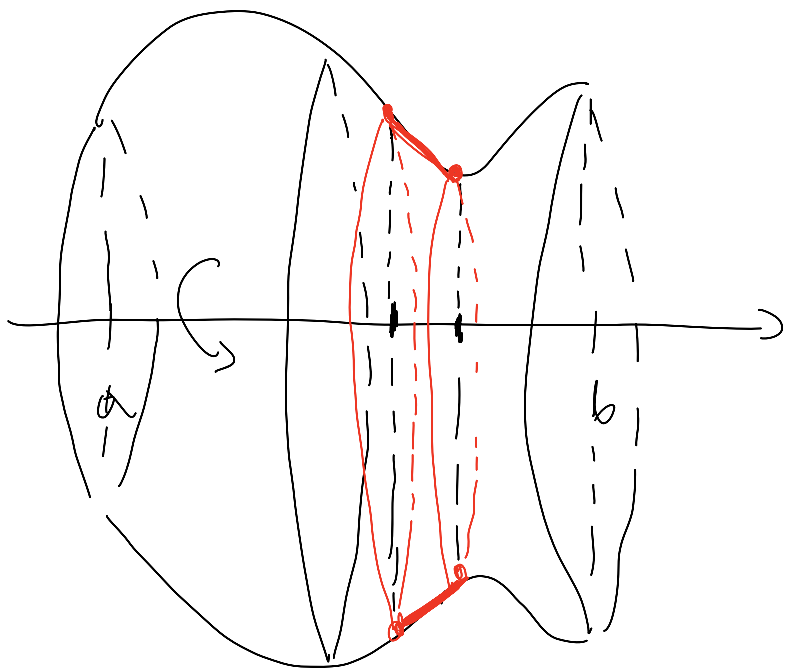
$$V = \pi \int_{-r}^{r} \left(\sqrt{r^2 - x^2}\right)^2 dx = \pi \int_{-r}^{r} \left(r^2 - x^2\right) dx =$$

$$= \pi r^2 \int_{-r}^{r} dx - \pi \int_{-r}^{r} x^2 \, dx =$$

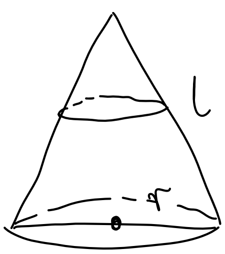
$$= \pi r^2 \, 2r - \pi \frac{x^3}{3} \Big|_{-r}^{r} =$$

$$= 2\pi r^3 - \pi \left[\frac{r^3}{3} - \frac{(-r)^3}{3} \right] =$$

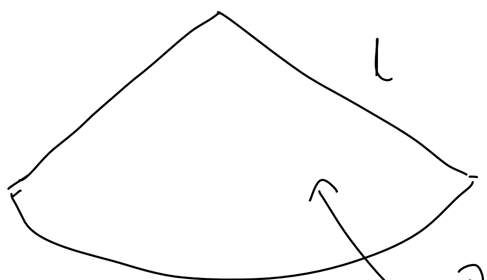
$$= 2\pi r^3 - \pi \frac{2r^3}{3} = \boxed{\frac{4}{3}\pi r^3}$$



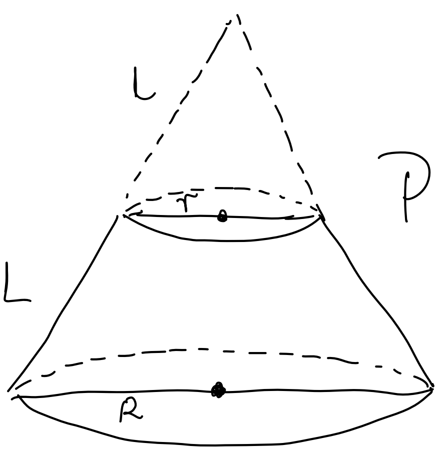
$$P = \sum_{i=1}^{n} P_i \xrightarrow[\delta \to 0]{n \to +\infty} 2\pi \int_a^b f(x)\sqrt{1+(f'(x))^2}\,dx$$



$$P = \pi r l$$



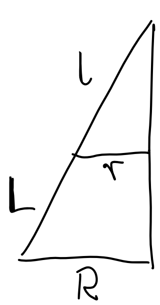
$$P = \frac{2\pi r}{2\pi l}\cdot \pi l^2 = \pi r l$$



$$P = \pi R(L+l) - \pi r l = \pi\left[RL + \frac{LRr}{R-r} - \frac{Lr^2}{R-r}\right] =$$

$$= \pi L \frac{R^2 - \cancel{Rr} + \cancel{Rr} - r^2}{R-r} =$$

$$= \pi L \frac{(R-r)(R+r)}{R-r} = \boxed{\pi L(R+r)}$$



$$\frac{r}{l} = \frac{R}{L+l}$$

$$lR = Lr + lr$$

$$l = \frac{Lr}{R-r}$$

Pole pow. bocznej stożka ściętego

$$\sum_{i=1}^{n} P_i = \sum_{i=1}^{n} \pi \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2} \cdot \left(f(x_i) + f(x_{i-1})\right)$$

$$= \pi \sum_{i=1}^{n} \Delta x_i \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2} \cdot \left(f(x_i) + f(x_{i-1})\right)$$

tw. Lagrange'a

$$= \pi \sum_{i=1}^{n} \Delta x_i \sqrt{1 + (f'(c_i))^2} \left(f(x_i) + f(x_{i-1})\right)$$

$$= 2\pi \sum_{i=1}^{n} \Delta x_i \sqrt{1 + (f'(c_i))^2} \left(f(c_i) + \frac{f(x_i) + f(x_{i-1})}{2} - f(c_i) \right)$$

$$= \underbrace{2\pi \sum_{i=1}^{n} \Delta x_i \sqrt{1 + (f'(c_i))^2} \cdot f(c_i)}_{\substack{\downarrow \; n \to +\infty \\ \delta \to 0 \\ 2\pi \int_a^b f(x) \sqrt{1 + (f'(x))^2}\, dx}} + \underbrace{2\pi \sum_{i=1}^{n} \Delta x_i \sqrt{1 + (f'(c_i))^2} \cdot \underbrace{\left(\frac{f(x_i) + f(x_{i-1})}{2} - f(c_i) \right)}_{\text{„małe"}}}_{\substack{\downarrow \; n \to +\infty \\ \delta \to 0 \\ 0}}$$

0 (ale wymaga to drobnego uzasadnienia, musimy wykorzystać ciągłość funkcji f)

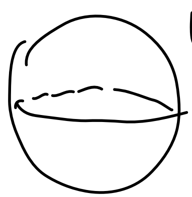
Pole powierzchni bryły obrotowej

Jeżeli funkcja f jest ciągła i nieujemna na przedziale $[a, b]$, to polem powierzchni bocznej bryły obrotowej

$$\{(x, y, z) \in \mathbb{R}^3 \colon x \in [a, b],\ y^2 + z^2 \leqslant f^2(x)\}$$

nazywamy liczbę

$$2\pi \int_a^b f(x)\sqrt{1 + (f'(x))^2}\, dx.$$



$\leftarrow \sqrt{r^2 - x^2}$

$$P = 2\pi \int_{-r}^{r} \cancel{\sqrt{r^2 - x^2}} \cdot \frac{r}{\cancel{\sqrt{r^2 - x^2}}}\, dx = 2\pi r \cdot \int_{-r}^{r} dx =$$

$$= 2\pi r \cdot 2r = \boxed{4\pi r^2}$$