Pochodne wyższych rzędów

$$f : (a, b) \to \mathbb{R}$$

$$f' : (a, b) \to \mathbb{R}$$

$$(f')' = f'' : (a, b) \to \mathbb{R}$$

$$(f'')' = f''' : (a, b) \to \mathbb{R}$$

$$(f''')' = f^{(4)} : (a, b) \to \mathbb{R}$$

$$\vdots$$

$$(f^{(n)})' = f^{(n+1)} : (a, b) \to \mathbb{R}$$

$$f(x) = x^4, \quad x \in \mathbb{R}$$

$$f'(x) = 4x^3$$

$$f''(x) = 4 \cdot 3x^2 = 12x^2$$

$$f'''(x) = 24x$$

$$f^{(4)}(x) = 24$$

$$f^{(n)}(x) = 0, \quad n \geqslant 5$$

Pochodne wyższych rzędów

Określoną indukcyjnie liczbę

$$f^{(n)}(x_0) = \begin{cases} f(x_0), & n = 0, \\ \left(f^{(n-1)}\right)'(x_0), & n \geqslant 1, \end{cases}$$

o ile istnieje, nazywamy **pochodną n-tego rzędu** funkcji f w punkcie x_0.

$f(x) = e^x, \quad x \in \mathbb{R}$

$f^{(n)}(x) = e^x, \quad x \in \mathbb{R}$

$f(x) = \sin x, \quad x \in \mathbb{R}$

$f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$f^{(4)}(x) = \sin x$

$f^{(5)}(x) = \cos x$

$\vdots$

$f^{(2n)}(x) = (-1)^n \sin x$

$f^{(2n+1)}(x) = (-1)^n \cos x$

$$(f\cdot g)' = f'g + fg'$$

$$(f\cdot g)'' = (f'g + fg')' = f''g + f'g' + f'g' + fg'' =$$

$$= f''g^{(0)} + 2f'g' + f^{(0)}g''$$

$$(a+b)^2 = a^2 + 2ab + b^2 =$$

$$= a^2b^0 + 2a^1b^1 + a^0b^2$$

$$(f\cdot g)''' = f'''\cdot g + 3f''g' + 3f'g'' + fg'''$$

$$(a+b)^3 = a^3b^0 + 3a^2b^1 + 3a^1b^2 + a^0b^3$$

$$\boxed{(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k}$$

$$\Downarrow ?$$

$$(f\cdot g)^{(n)} = ?$$

Pochodna n-tego rzędu iloczynu

Wzór Leibniza

Jeżeli funkcje f i g mają pochodne n-tego rzędu w punkcie x_0, to

$$(f \cdot g)^{(n)}(x_0) = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)}(x_0) \cdot g^{(k)}(x_0)$$

$$(f \cdot g)^{(n)} = \sum_{k=0}^{n} \binom{n}{k} f^{(n-k)} \cdot g^{(k)}$$

Dow. Indukcja względem n.

$\ln 2 = ?$ $\quad \sqrt{5} = ?$ $\quad \sin 1 = ?$

$$w(x) = \underset{a_2}{2}x^2 + \underset{a_1}{3}x \underset{a_0}{-1}$$

$$w(0) = -1 = a_0$$

$$w'(x) = 4x + 3$$

$$w'(0) = 3 = a_1$$

$$w''(x) = 4$$

$$w''(0) = 4 = a_2 \cdot 2$$

$$w(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

$$w(0) = a_0$$

$$w'(x) = n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + 2a_2 x + a_1$$

$$w'(0) = a_1$$

$$w''(x) = n(n-1)a_n x^{n-2} + (n-1)(n-2)a_{n-1}x^{n-3} + \ldots + 2a_2$$

$$w''(0) = 2a_2 = 2!\,a_2$$

$$w'''(0) = 6a_3 = 3!\,a_3$$

$$w^{(4)}(0) = 24a_4 = 4!\,a_4$$

$$\vdots$$

$$w^{(k)}(0) = k!\,a_k$$

$$\vdots$$

$$w^{(n)}(0) = n!\,a_n$$

$$\boxed{\begin{aligned} w(x) &= a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0 = \\ &= \frac{w^{(n)}(0)}{n!}x^n + \frac{w^{(n-1)}(0)}{(n-1)!} + \ldots + \frac{w'(0)}{1!} + \frac{w(0)}{0!} \\ &= \boxed{\sum_{k=0}^{n} \frac{w^{(k)}(0)}{k!}x^k} \end{aligned}}$$

$$w(x) = \sum_{k=0}^{n} \frac{w^{(k)}(0)}{k!} x^k = w(0) + w'(0)x + \frac{w''(0)}{2!}x^2 + \dots + \frac{w^{(n)}(0)}{n!}x^n$$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k + ?$$

$\frac{f^{(k)}(0)}{k!} \in \mathbb{R}$

$\underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k}_{\text{wielomian}}$

$$\boxed{f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k}$$

Reszta

$f:(a,b) \to \mathbb{R}$, $x_0 \in (a,b)$

$$v(x) = w(x + x_0)$$

$$v^{(k)}(x) = w^{(k)}(x+x_0) \cdot \underbrace{(x+x_0)'}_{1}$$

$$w(x) = \sum_{k=0}^{n} \frac{w^{(k)}(0)}{k!} x^k$$

$$v(x) = \sum_{k=0}^{n} \frac{v^{(k)}(0)}{k!} x^k = \sum_{k=0}^{n} \frac{w^{(k)}(x_0)}{k!} x^k$$

$$w(x+x_0) = \sum_{k=0}^{n} \frac{w^{(k)}(x_0)}{k!} x^k$$

$$\boxed{\begin{aligned} t &:= x + x_0 \\ t - x_0 &= x \end{aligned}}$$

$$w(t) = \sum_{k=0}^{n} \frac{w^{(k)}(x_0)}{k!} (t - x_0)^k, \quad t \in \mathbb{R}$$

$$\boxed{w(x) = \sum_{k=0}^{n} \frac{w^{(k)}(x_0)}{k!} (x - x_0)^k}$$

$$\boxed{w(x) = \sum_{k=0}^{n} \frac{w^{(k)}(x_0)}{k!}(x-x_0)^k}, \quad x, x_0 \in \mathbb{R}$$

$$f:(a,b) \to \mathbb{R}, \quad x, x_0 \in (a,b)$$

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{\text{wielomian zmiennej } x} + \boxed{\begin{matrix} ? \\ \text{Reszta} \end{matrix}}$$

$$R(x) := f(x) - \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \dots}$$

Niech

$$g(t) = f(x) + \\ - \left[f(t) + f'(t)(x-t) + \frac{f''(t)}{2!}(x-t)^2 + \dots + \frac{f^{(n)}(t)}{n!}(x-t)^n \right]$$

$$g'(t) = 0 - \left[\cancel{f'(t)} + \cancel{f''(t)(x-t)} - \cancel{f'(t)} + \cancel{\frac{f'''(t)}{2!}(x-t)^2} - \cancel{\frac{f''(t)}{2!}2(x-t)} + \dots + \frac{f^{(n+1)}(t)}{n!}(x-t)^n - \cancel{\frac{f^{(n)}(t)}{n!}n(x-t)^{n-1}} \right]$$

$$= -\frac{f^{(n+1)}(t)}{n!}(x-t)^n$$

$$\boxed{g'(t) = -\frac{f^{(n+1)}(t)}{n!}(x-t)^n} \quad (*)$$

Ponadto

$g(x) = 0$,

$g(x_0) = R(x)$.

Niech
$$h(t) = g(t) - \frac{1}{(x-x_0)^{n+1}} R(x)(x-t)^{n+1}.$$

Mamy
$$h(x) = g(t) - 0 = 0$$
oraz
$$h(x_0) = g(x_0) - \frac{1}{\cancel{(x-x_0)^{n+1}}} R(x) \cancel{(x-x_0)^{n+1}} = R(x) - R(x) = 0.$$

Z tw. Rolle'a zastosowanego do funkcji h wynika istnienie punktu c leżącego między x a x_0, dla którego
$$h'(c) = 0.$$

Ponieważ
$$h'(t) = g'(t) + \frac{n+1}{(x-x_0)^{n+1}} R(x)(x-t)^n,$$
to
$$\frac{n+1}{(x-x_0)^{n+1}} R(x) \cancel{(x-c)^n} = -g'(c) \overset{(*)}{=} \frac{f^{(n+1)}(c)}{n!} \cancel{(x-c)^n}.$$

$$\Longrightarrow \boxed{R(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.}$$

Wzór Taylora

Załóżmy, że $I = [a, b]$ jest przedziałem domkniętym oraz $x, x_0 \in I$, $x \neq x_0$. Jeżeli dla liczby naturalnej $n \geqslant 1$ funkcja f ma

- ciągłą pochodną rzędu n na przedziale I,
- pochodną rzędu $n+1$ na przedziale (a, b),

to istnieje taki punkt c, leżący między x a x_0, że

$$f(x) = \underbrace{\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k}_{\text{Wielomian Taylora}} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}}_{\text{Reszta zapisana w postaci Lagrange'a.}}.$$

Príklad. $\sqrt{3.96} \sim 2$

$$\boxed{f(x) = \sqrt{x}, \quad x = 3.96, \quad x_0 = 4}$$

$f(3.96) = ?$

$$f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-\frac{1}{2}}$$

$$f''(x) = -\frac{1}{4} x^{-3/2}$$

$$f'''(x) = \frac{3}{8} x^{-5/2}$$

$$f^{(4)}(x) = -\frac{15}{16} x^{-7/2}$$

$$+ \boxed{\frac{f^{(4)}(c)}{4!}(x-x_0)^4}$$

$$f(x) = \underline{f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \frac{f'''(x_0)}{3!}(x-x_0)^3}$$

$$\sqrt{3.96} \cong \sqrt{4} + \frac{1}{2\sqrt{4}}(3.96-4) - \frac{\frac{1}{4} 4^{-3/2}}{2!}(3.96-4)^2 + \frac{\frac{3}{8} 4^{-5/2}}{3!}(3.96-4)^3$$

$$= 2 - \frac{1}{4} \cdot \frac{4}{100} - \frac{1}{4 \cdot \cancel{8} \cdot \cancel{8}_1} \frac{\cancel{4}^{\cancel{2}}}{100^2} - \frac{\cancel{3}}{8} \cdot \frac{1}{\cancel{6}_{\cancel{2}}} \cdot \frac{4}{\cancel{32}} \cdot \frac{\cancel{4}^{\cancel{3}}}{100^3} =$$

$$= 2 - \frac{1}{100} - \underbrace{\frac{1}{4} \cdot \frac{1}{10000}}_{0.000025} - \underbrace{\frac{1}{8} \cdot \frac{1}{1000000}}_{0.000000125} =$$

$$= \boxed{1.989974875} \qquad \boxed{1.98997487 4 \ldots}$$

$$R(3.96) = \frac{f^{(4)}(c)}{4!}\left(-\frac{4}{100}\right)^4 = \frac{1}{\cancel{4!}_{\cancel{6}\,\cancel{3}}}\left(-\frac{\cancel{15}^{\,5}}{\cancel{16}}c^{-7/2}\right)\frac{2\,\cancel{4^{4}}\,\cancel{2}}{100^4} =$$

$$= -10\,c^{-7/2}\cdot\frac{1}{100^4}$$

$$c \in (3.96,\ 4)$$

$$\Longrightarrow\quad |R(3.96)| = \frac{1}{c^{7/2}\cdot 10^7} \leqslant \frac{1}{3^3\cdot 10^7} \leqslant \frac{1}{10^8}$$

Można to oczywiście lepiej oszacować.