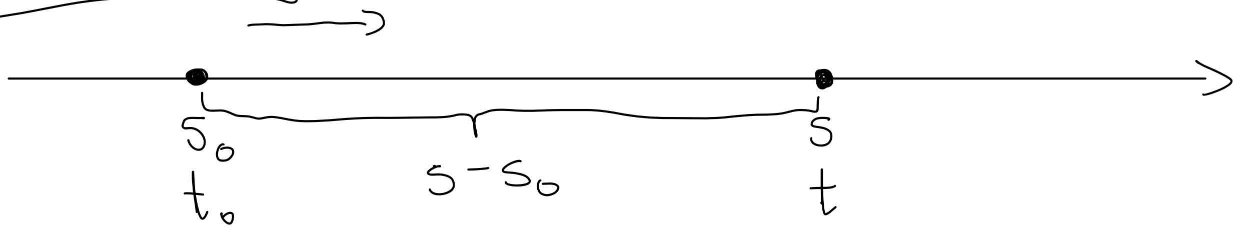
Rachunek różniczkowy

Prędkość?



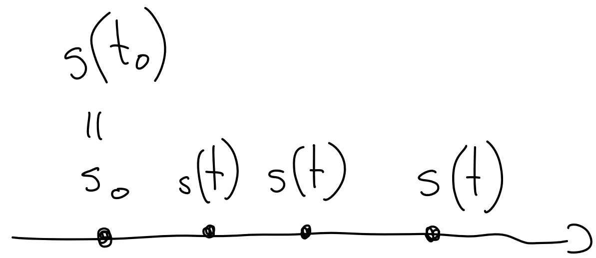
$$v = \frac{s}{t}$$

droga

czas

prędkość

$$v_{\text{średnia}} = \frac{s-s_0}{t-t_0}$$

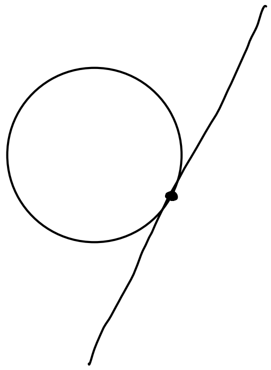


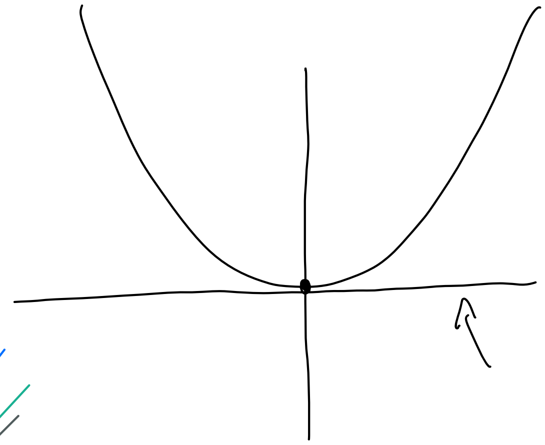
$$v_{\text{chwilowa}} = ?$$

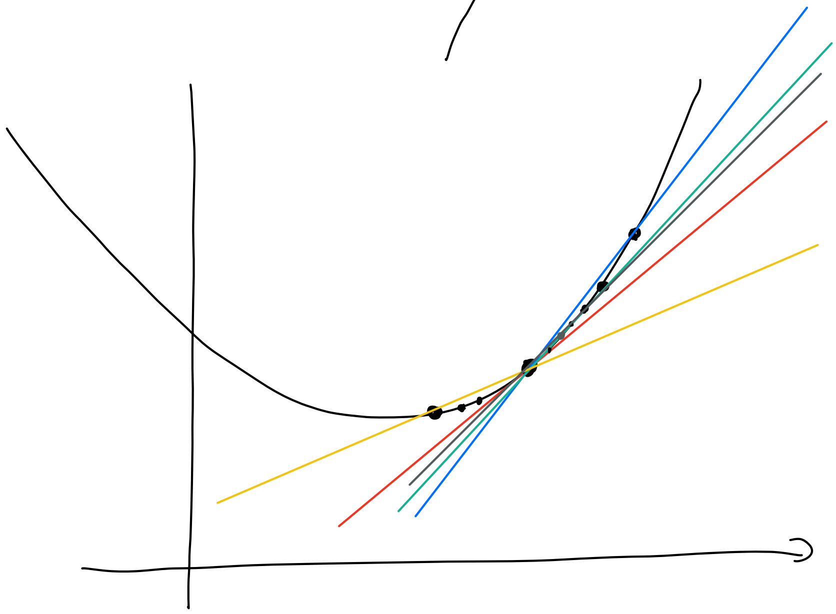
$$v_{\text{śr}} = v_{\text{śr}}(t) = \frac{s(t)-s(t_0)}{t-t_0} \quad \leftarrow \quad \text{dla } t=t_0$$

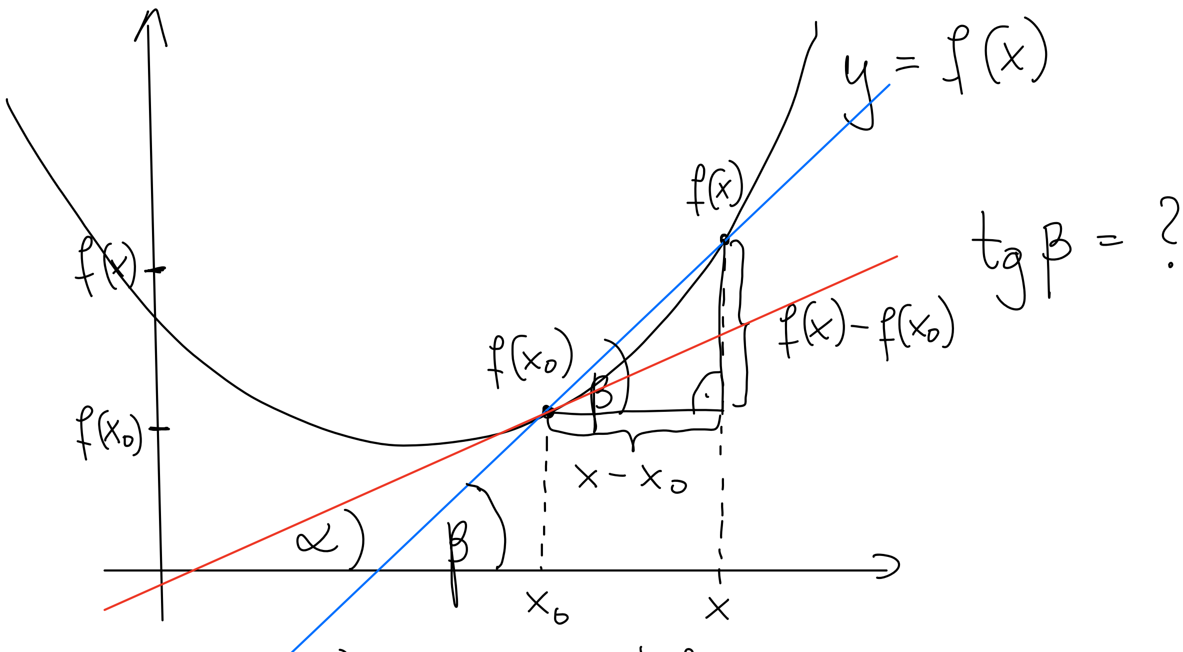
$$\boxed{v_{\text{chwilowa}} \overset{\text{def.}}{=} \lim_{t\to t_0} v_{\text{śr}}(t) = \lim_{t\to t_0} \frac{s(t)-s(t_0)}{t-t_0}}$$

Styczna?









$$\operatorname{tg}\beta = \frac{f(x)-f(x_0)}{x-x_0} \qquad \alpha \stackrel{def}{=} \lim_{x\to x_0} \beta$$

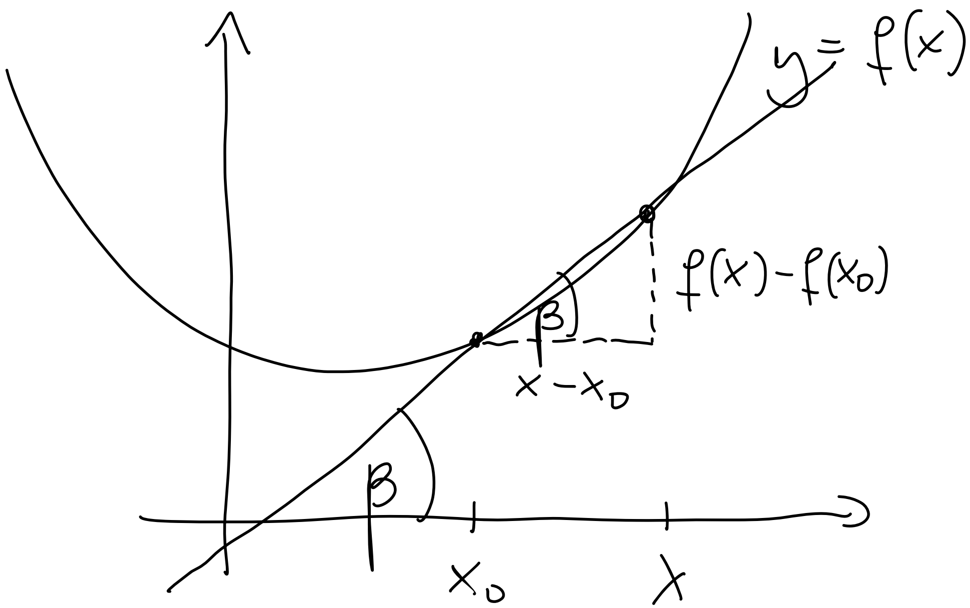
$$\boxed{\operatorname{tg}\alpha \stackrel{def.}{=} \lim_{x\to x_0} \operatorname{tg}\beta(x) = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}}$$

Iloraz różnicowy

Niech $f : (a, b) \to \mathbb{R}$ oraz $x, x_0 \in (a, b)$, $x \neq x_0$. Liczbę

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{\Delta f}{\Delta x}$$

nazywamy **ilorazem różnicowym** funkcji f w punkcie x_0 dla przyrostu $x - x_0$.



$$\operatorname{tg} \beta = \frac{f(x) - f(x_0)}{x - x_0}$$

$x - x_0 = \Delta x$

$f(x) - f(x_0) = \Delta y = \Delta f$

Pochodna funkcji

Niech $f\colon (a,b) \to \mathbb{R}$ oraz $x_0 \in (a,b)$. Jeżeli istnieje (właściwa) granica

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

to nazywamy ją **pochodną funkcji** f w punkcie x_0 i oznaczamy

$$f'(x_0).$$

Mówimy wtedy, że funkcja f jest **różniczkowalna** w punkcie x_0.

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \left\{ \begin{array}{l} h = x - x_0 \\ x = x_0 + h \\ x \to x_0 \equiv h \to 0 \end{array} \right\} = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} =$$

$$= \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

$$f'(x_0) \qquad \frac{df}{dx}(x_0) \qquad \dot{f}(x_0) \qquad Df(x_0)$$

Przykłady.

1) $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^2$ $\quad x_0 \in \mathbb{R}$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} \frac{\cancel{(x - x_0)}(x + x_0)}{\cancel{x - x_0}} = 2x_0$$

$$f'(x_0) = 2x_0$$

$$(x^2)' = 2x \leftarrow$$

2) $f: \mathbb{R} \to \mathbb{R}$ $\quad f(x) = x^n$, $\quad n \in \mathbb{N}$, $\quad x_0 \in \mathbb{R}$

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{x^n - x_0^n}{x - x_0} = \frac{1}{\cancel{x - x_0}} \overbrace{\cancel{(x - x_0)}(x^{n-1} + x^{n-2} x_0 + \ldots + x^1 x_0^{n-2} + x_0^{n-1})}^{x^n - x_0^n}$$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \underbrace{(x^{n-1} + \ldots + x_0^{n-1})}_{n} = n x_0^{n-1}$$

$$(x^n)' = n x^{n-1}$$

3) (ćw) $\quad f(x) = x^a$, $\quad$ •) $a \in \mathbb{Q}$, $\quad$ •) $a \in \mathbb{R}$

4) $f: \mathbb{R} \to \mathbb{R}$, $\quad f(x) = \sin x$, $\quad x_0 \in \mathbb{R}$

$$\frac{f(x) - f(x_0)}{x - x_0} = \frac{\sin x - \sin x_0}{x - x_0} = \frac{2 \sin \frac{x - x_0}{2} \cos \frac{x + x_0}{2}}{x - x_0}$$

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{\sin \frac{x - x_0}{2}}{\frac{x - x_0}{2}} \cdot \underbrace{\cos \frac{x + x_0}{2}}_{\downarrow x \to x_0 \atop \cos x_0} = \left| \begin{array}{l} h = \frac{x - x_0}{2} \\ x = 2h + x_0 \end{array} \right|$$

$$= \lim_{h \to 0} \underbrace{\frac{\sin h}{h}}_{\downarrow h \to 0 \atop 1} \cdot \underbrace{\cos \frac{2h + 2x_0}{2}}_{\downarrow h \to 0 \atop \cos x_0} = \cos x_0$$

$$\boxed{(\sin x)' = \cos x}$$

5) $f:(0,+\infty)\to\mathbb{R}$, $f(x)=\ln x$, $x_0\in(0,+\infty)$

$$\lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}=\lim_{h\to 0}\frac{\ln(x_0+h)-\ln x_0}{h}=$$

$$=\lim_{h\to 0}\frac{\ln\frac{x_0+h}{x_0}}{h}=$$

$$=\lim_{h\to 0}\frac{1}{h}\ln\left(\frac{x_0+h}{x_0}\right)=$$

$$=\lim_{h\to 0}\frac{1}{h}\ln\left(1+\frac{h}{x_0}\right)=$$

$$=\lim_{h\to 0}\ln\left[\left(1+\frac{h}{x_0}\right)^{\frac{1}{h}}\right]=$$

$$=\lim_{h\to 0}\ln\left\{\underbrace{\left[\left(1+\frac{h}{x_0}\right)^{\frac{x_0}{h}}\right]}_{\downarrow h\to 0 \;\; e}{}^{\frac{1}{x_0}}\right\}=$$

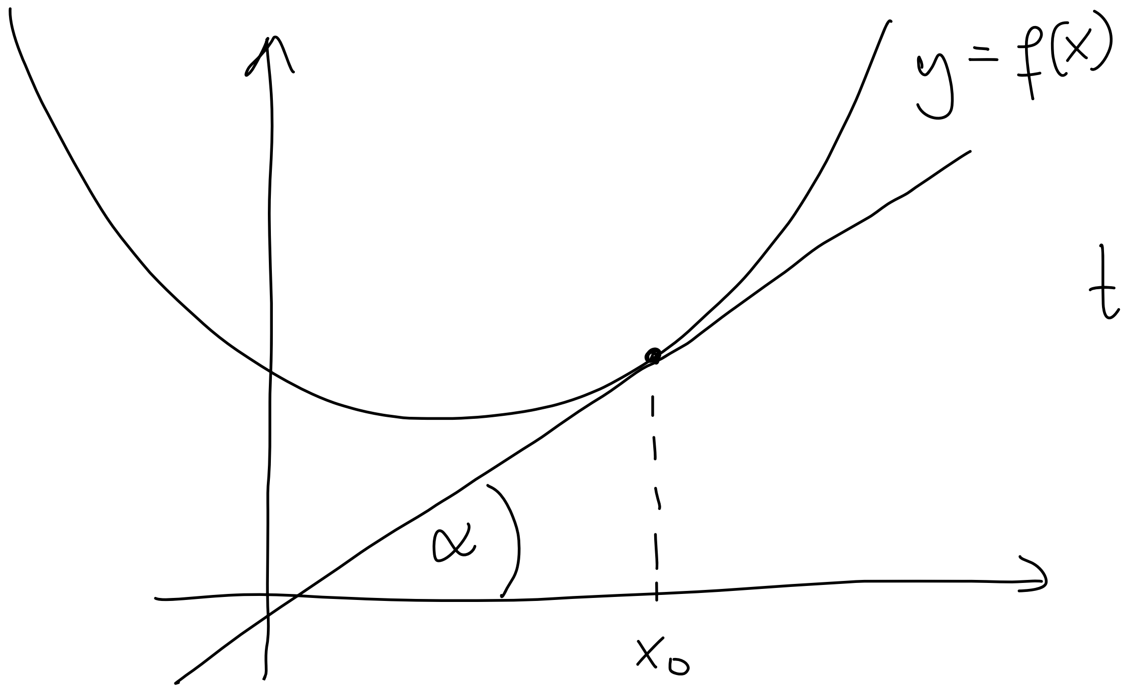
$$\left|\begin{array}{c}\text{ciągłość}\\ \text{funkcji logarytmicznej}\end{array}\right|=\ln\left(\lim_{h\to 0}\left\{\left[\left(1+\frac{h}{x_0}\right)^{\frac{x_0}{h}}\right]^{\frac{1}{x_0}}\right\}\right)=$$

$$=\ln\left(e^{\frac{1}{x_0}}\right)=\frac{1}{x_0}$$

$$(\ln x)'=\frac{1}{x}$$

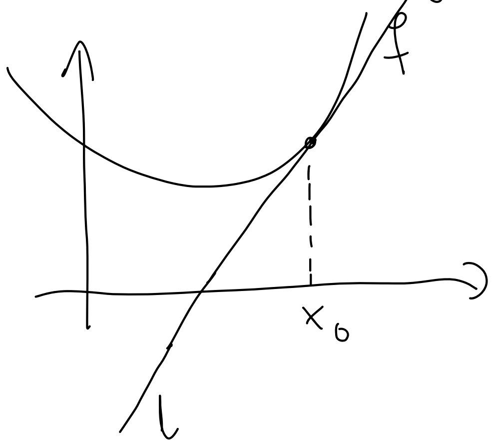
Styczna

Jeżeli funkcja f jest różniczkowalna w punkcie x_0, to **styczną** do wykresu funkcji f w punkcie x_0 nazywamy prostą przechodzącą przez $f(x_0)$ o współczynniku kierunkowym równym $f'(x_0)$.



$$\operatorname{tg}\alpha = f'(x_0)$$

Jakim wzorem dana jest styczna?



$$l: \quad y = ax + b$$

1) $f(x_0) = a x_0 + b \quad \Rightarrow \quad b = f(x_0) - a x_0$

2) $a = f'(x_0)$

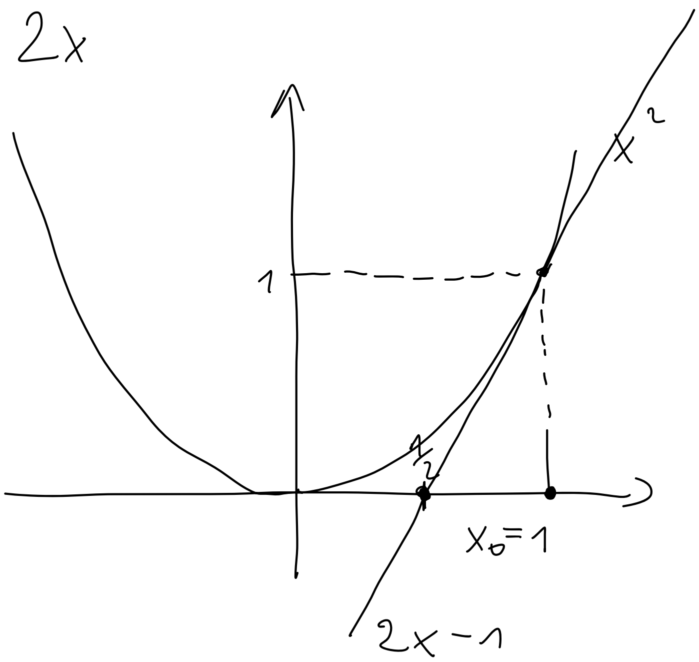
$$1)+2) \Rightarrow b = f(x_0) - f'(x_0)\, x_0$$

$$l: \quad y = f'(x_0)\, x + f(x_0) - f'(x_0)\, x_0 =$$

$$= \boxed{f(x_0) + f'(x_0)(x - x_0)}$$

$$(x^2)' = 2x$$

$$l: \quad f(x_0) + f'(x_0)(x - x_0) = y$$



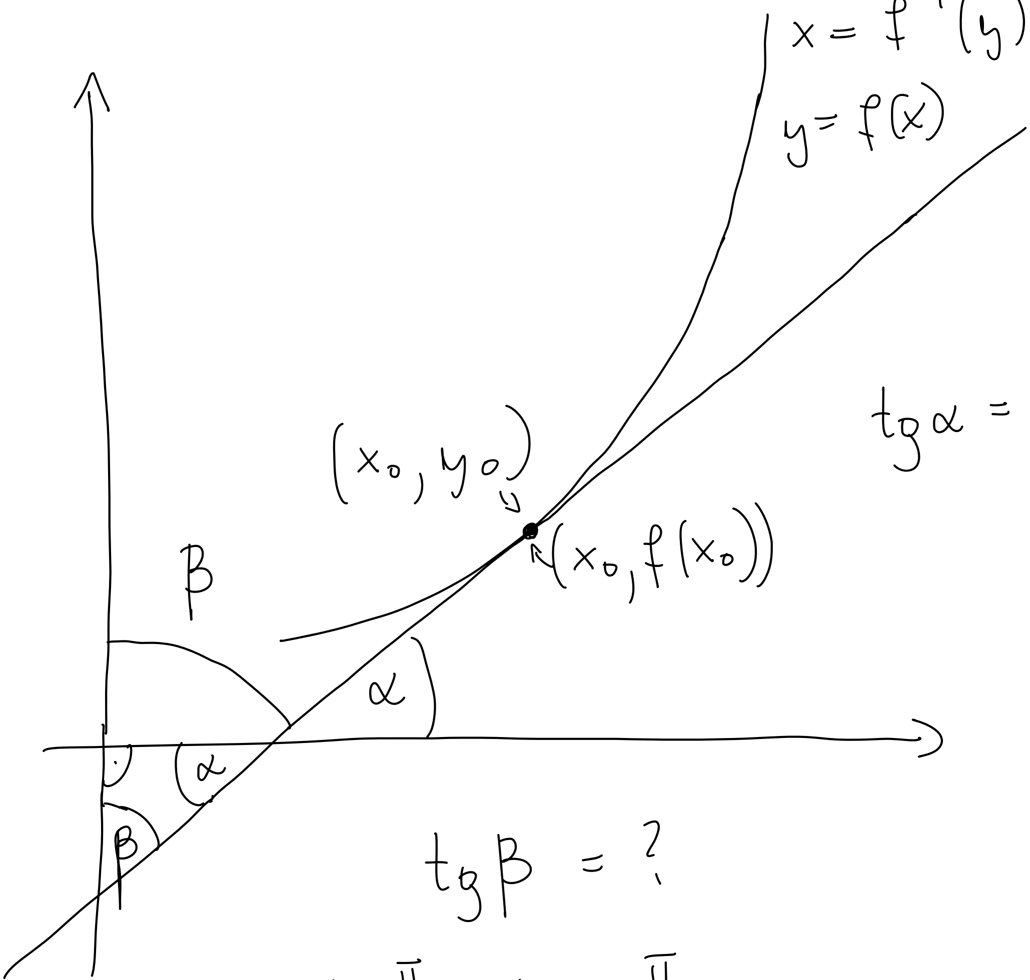
$$y = 1^2 + 2 \cdot 1 \cdot (x - 1)$$

$$\boxed{y = 2x - 1}$$

$$\underbrace{f, \quad f^{-1} \text{ istnieje}, \quad f'(x_0),}$$

$$\Downarrow$$

$$(f^{-1})'(y_0) = \,?$$



$$\operatorname{tg}\alpha = f'(x_0)$$

$$\operatorname{tg}\beta = \,?$$

$$\beta = \pi - \frac{\pi}{2} - \alpha = \frac{\pi}{2} - \alpha$$

$$\operatorname{tg}\beta = \operatorname{tg}\left(\frac{\pi}{2} - \alpha\right) = \operatorname{ctg}\alpha = \frac{1}{\operatorname{tg}\alpha}$$

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)}$$

$$(f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}$$

Pochodna funkcji odwrotnej

gwarantuje odwracalność

Twierdzenie

Jeżeli funkcja f określona w przedziale (a, b) jest ciągła i ściśle monotoniczna oraz $f'(x_0) \neq 0$, to funkcja odwrotna f^{-1} jest różniczkowalna w punkcie $y_0 = f(x_0)$ oraz

$$(f^{-1})'(y_0) = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}.$$

$y_0 = f(x_0)$

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{y \to y_0} \frac{1}{\frac{y - y_0}{f^{-1}(y) - f^{-1}(y_0)}} =$$

$$= \left| \begin{array}{l} x = f^{-1}(y) \\ x_0 = f^{-1}(y_0) \end{array} \right| = \lim_{x \to x_0} \frac{1}{\frac{f(x) - f(x_0)}{x - x_0}} = \frac{1}{f'(x_0)}$$

1) $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$ (ch.)

2) $g : [-1,1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ $g(x) = \arcsin x$

$(\arcsin x)' = ?$

$g = f^{-1}$, $f : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1,1]$, $f(x) = \sin x$

$f'(x) = (\sin x) = \cos x$

$$g'(y_0) = (f^{-1})'(y_0) = \frac{1}{f'(f^{-1}(y_0))}$$

$$g'(y_0) = \frac{1}{\cos(\arcsin y_0)} = \frac{1}{\sqrt{1 - \sin^2(\underbrace{\arcsin y_0}_{\in [-\frac{\pi}{2}, \frac{\pi}{2}]})}} =$$

$$\left\{ \begin{array}{l} \cos^2\alpha = 1 - \sin^2\alpha \\ |\cos\alpha| = \sqrt{1 - \sin^2\alpha} \\ \cos\alpha = \sqrt{1-\sin^2\alpha}, \\ \text{o ile } \alpha \in [-\frac{\pi}{2}, \frac{\pi}{2}] \end{array} \right\} = \frac{1}{\sqrt{1 - y_0^2}}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \quad x \in (-1,1)$$

Pochodne funkcji elementarnych

- $(c)' = 0$
- $(x^a)' = ax^{a-1}$
- $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$
- $(\frac{1}{x})' = -\frac{1}{x^2}$
- $(\mathrm{e}^x)' = \mathrm{e}^x$
- $(a^x)' = a^x \ln a$
- $(\ln x)' = \frac{1}{x}$
- $(\log_a x)' = \frac{1}{x \ln a}$
- $(\sin x)' = \cos x$
- $(\cos x)' = -\sin x$
- $(\operatorname{tg} x)' = \frac{1}{\cos^2 x}$
- $(\operatorname{ctg} x)' = -\frac{1}{\sin^2 x}$
- $(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$
- $(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$
- $(\operatorname{arc\,tg} x)' = \frac{1}{1+x^2}$
- $(\operatorname{arc\,ctg} x)' = -\frac{1}{1+x^2}$

Algebraiczne własności pochodnej

Twierdzenie

Jeżeli funkcje f i g są różniczkowalne w punkcie x_0, to

- $(c \cdot f)'(x_0) = c \cdot f'(x_0)$ dla dowolnego $c \in \mathbb{R}$,
- $(f + g)'(x_0) = f'(x_0) + g'(x_0)$,
- $(f \cdot g)'(x_0) = f'(x_0)g(x_0) + f(x_0) \cdot g'(x_0)$,
- $\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g^2(x_0)}$.