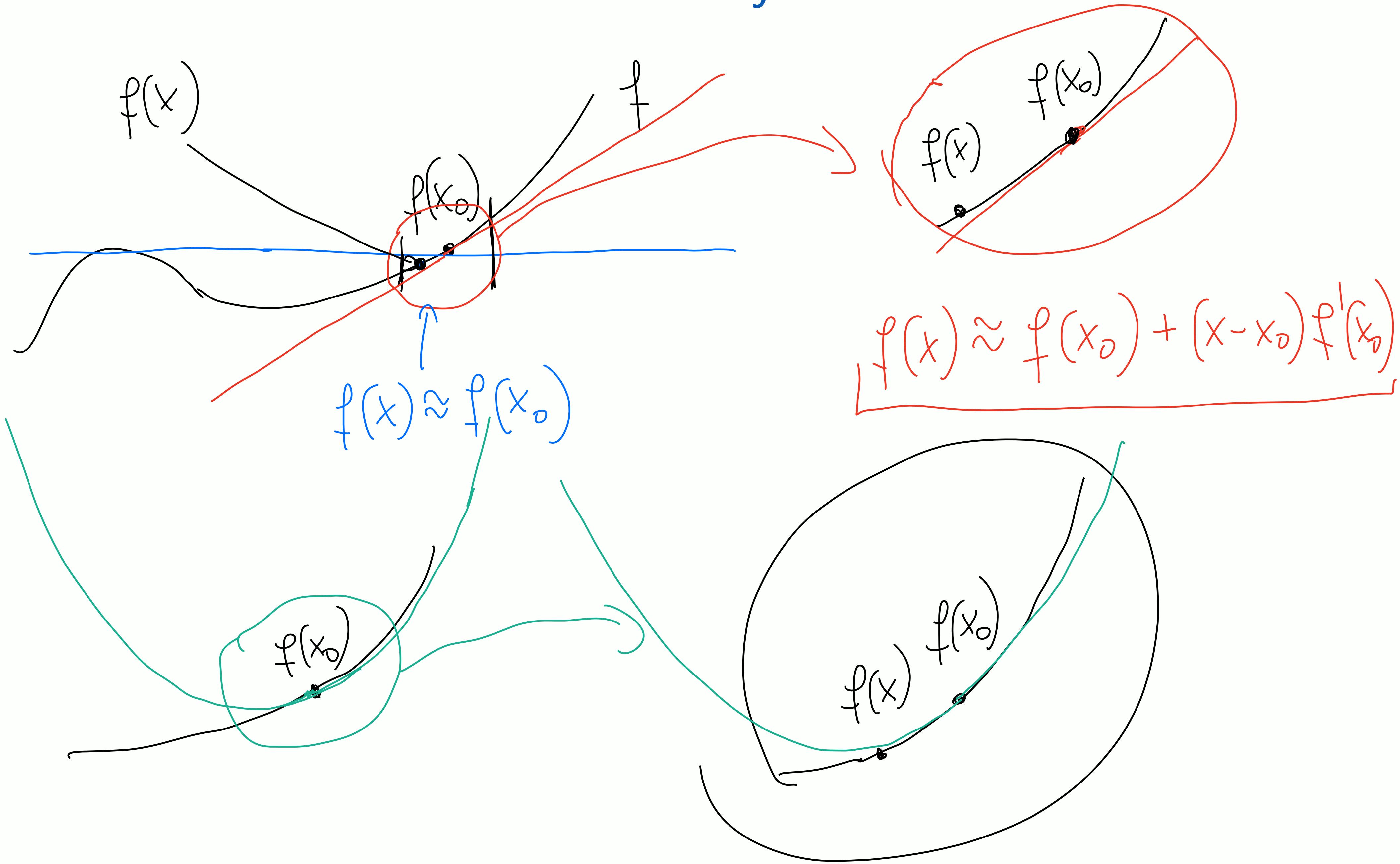
Wzór Taylora



$$f(x) \approx f(x_0)$$

$$f(x) \approx f(x_0) + (x - x_0) f'(x_0)$$

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$$

$$f(0) = a_0$$

$$f'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \dots + 2 a_2 x + a_1$$

$$f'(0) = a_1$$

$$f''(x) = n(n-1) a_n x^{n-2} + (n-1)(n-2) a_{n-1} x^{n-3} + \dots + 2 a_2$$

$$f''(0) = 2 a_2$$

$$f'''(x) = n(n-1)(n-2) a_n x^{n-3} + \dots + 3 \cdot 2 \cdot a_3$$

$$f'''(0) = 3 \cdot 2 \cdot a_3$$

$$a_0 = f(0), \quad a_1 = f'(0), \quad a_2 = \frac{1}{2} f''(0), \quad a_3 = \frac{1}{2 \cdot 3} f'''(0)$$

$$\boxed{a_k = \frac{f^{(k)}(0)}{k!}}, \quad k = 0, 1, \dots, n$$

$$f(x) = \boxed{\underbrace{f(0)}_{a_0} + \underbrace{f'(0)}_{a_1} x + \underbrace{\frac{f''(0)}{2}}_{a_2} x^2 + \dots + \underbrace{\frac{f^{(n)}(0)}{n!}}_{a_n} x^n}$$

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k$$

Wzór Taylora

Załóżmy, że $I = \langle a, b \rangle$ jest przedziałem domkniętym oraz $x, x_0 \in I$, $x \neq x_0$. Jeżeli dla liczby naturalnej $n \geqslant 1$ funkcja f ma

- ⇝ ciągłą pochodną rzędu $n-1$ na przedziale I,
- ⇝ pochodną rzędu n na przedziale (a, b),

f ma dużo pochodnych

to istnieje taki punkt c, leżący między x a x_0, że

$$f(x) = \underbrace{\sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k}_{\text{wielomian Taylora}} + \underbrace{\frac{f^{(n)}(c)}{n!}(x - x_0)^n}_{\text{reszta (Lagrange'a)}}.$$

$$f(x) = \underbrace{\sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k}_{} + \underbrace{\frac{f^{(n)}(c)}{n!}(x-x_0)^n}_{}$$

$$A = \frac{1}{(x-x_0)^n}\left[f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k\right]$$

$$\boxed{g(t) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(t)}{k!}(x-t)^k - A(x-t)^n}$$

$$g(x) = f(x) - \underbrace{\frac{f^{(0)}(x)}{0!}}_{f(x)} - \sum_{k=1}^{n-1} \frac{f^{(k)}(x)}{k!}(x-x)^k - A(x-x)^n = 0$$

$$g(x_0) = f(x) - \sum_{k=0}^{n-1} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k - \boxed{A(x-x_0)^n} = 0$$

$$g(x) = g(x_0) = 0$$

$$g'(t) = \boxed{-\frac{(x-t)^{n-1}}{(n-1)!} f^{(n)}(t) \boxed{+ An(x-t)^{n-1}}}$$

Tw. Rolle'a $\Rightarrow$ istnieje c między x a x_0, dla którego $g'(c) = 0$.

$$\Updownarrow$$

$$An\cancel{(x-c)^{n-1}} = \frac{f^{(n)}(c)}{(n-1)!}\cancel{(x-c)^{n-1}}$$

$$\boxed{A = \frac{f^{(n)}(c)}{n!}}$$

Przykład

$\sqrt{3.96} \approx 2$ $\qquad$ $\sqrt{3.96} \approx 1.9\ldots ?$

$$\boxed{\sqrt{4} = 2}$$

$$f(x) = \sqrt{x}$$

$f(3.96) = ?$ $\qquad$ $f(4) = 2$

$f(x)=\sqrt{x}$, $f(3.96)=?$ $x_0=4$ $f(x_0)=2$

$$f'(x)=\frac{1}{2\sqrt{x}}=\frac{1}{2}x^{-1/2}$$

$$f''(x)=-\frac{1}{4}x^{-3/2}$$

$$f'''(x)=\frac{3}{8}x^{-5/2}$$

$$f^{(4)}(x)=-\frac{15}{16}x^{-7/2}$$

$$\boxed{\begin{array}{ll} f'(4)=\frac{1}{4} & f(4)=2 \\ f''(4)=-\frac{1}{4}\cdot\frac{1}{8} & \\ f'''(4)=\frac{3}{8}\cdot\frac{1}{32} & \\ f^{(4)}(4)=-\frac{15}{16}\cdot\frac{1}{128} & \end{array}}$$

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\dots+\frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1}$$

+ Reszta

$n=4$

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\frac{f'''(x_0)}{3!}(x-x_0)^3+R$$

$x_0=4$, $f(x)=\sqrt{x}$

$$\sqrt{x}=\underbrace{2+\frac{1}{4}\cdot(x-4)-\frac{1}{4}\cdot\frac{1}{8}\cdot\frac{1}{2}\cdot(x-4)^2+\frac{3}{8}\cdot\frac{1}{32}\cdot\frac{1}{6}(x-4)^3}_{\text{wielomian Taylora}}+R$$

$$\sqrt{3.96}\approx 2+\frac{1}{4}\cdot\frac{-4}{100}-\frac{1}{4}\cdot\left(\frac{1}{8}\cdot\frac{1}{2}\right)\cdot\left(\frac{4}{100}\right)^2+\frac{3}{8}\cdot\frac{1}{32}\cdot\frac{1}{6}\cdot\left(\frac{-4}{100}\right)^3=$$

$$=2-\frac{1}{100}-\frac{1}{4}\cdot\frac{1}{(100)^2}-\frac{1}{8}\cdot\frac{1}{100^3}=$$

$$=1.99-0.000025-0.000000125=$$

$$=1.989975-0.000000125=$$

$$=1.989974875$$

$$\sqrt{3.96}\approx 1.98997487421325\dots$$

$$\sqrt{3.96} \approx 1.98\ldots \quad + \underbrace{R}_{?}$$

$$R = \frac{f^{(n)}(c)}{n!}(x-x_0)^n$$

$$R = \frac{-\frac{15}{16}c^{-7/2}}{\boxed{4!}}\left(\frac{-4}{100}\right)^4 \qquad \underline{c \in (3.96,\ 4)}$$

$$= \boxed{-\frac{15}{16}\cdot\frac{1}{24}\cdot\frac{4^4}{100^4}}\cdot\boxed{\frac{1}{c^{7/2}}}_{<1}$$

$$|R| \le \boxed{\frac{15}{16}\cdot\frac{1}{24}\cdot\frac{4^4}{100^4}}\cdot 1 \approx \boxed{10^{-7}}$$

$$\frac{15}{16} \to 2 \to 1 \qquad \frac{4^4}{100^4} \to \frac{1}{25^4}$$

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)(x-x_0)^2}{2!} + \ldots + \frac{f^{(n-1)}(x_0)}{(n-1)!}(x-x_0)^{n-1} + \underline{R_n(x)}$$

Wzór Maclaurina = wzór Taylora dla $x_0 = 0$

$$f(x) = \underbrace{f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \ldots + \frac{f^{(n-1)}(0)}{(n-1)!}x^{n-1}}_{} + R_n(x)$$

1. $f(x) = e^x$, $\quad f^{(k)}(x) = e^x \quad f^{(k)}(0) = 1$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \ldots + \frac{1}{(n-1)!}x^{n-1} + \underbrace{\frac{e^c}{n!}x^n}_{R}$$

n duże $\Rightarrow R$ małe

2. $f(x) = \sin x \qquad f^{(k)}(x) = \begin{cases} \sin x, & k \equiv 0 \pmod 4 \\ \cos x, & k \equiv 1 \pmod 4 \\ -\sin x, & k \equiv 2 \pmod 4 \\ -\cos x; & k \equiv 3 \pmod 4 \end{cases}$

$$f^{(k)}(0) = \begin{cases} 0, & k \equiv 0 \pmod 2 \\ 1, & k \equiv 1 \pmod 4 \\ -1, & k \equiv 3 \pmod 4 \end{cases}$$

$$\sin x = \underbrace{x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \ldots \pm \frac{1}{(2n-1)!}x^{2n-1}} + R_n(x)$$

3.

$$\cos x = \underbrace{1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \ldots \pm \frac{1}{(2n)!}x^{2n}} + R_n(x)$$

$\ln(1+x)$, $\sqrt{1+x}$, $\operatorname{arctg} x$

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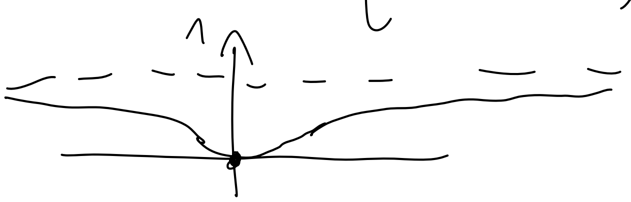
4. $f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases} \qquad \boxed{f^{(k)}(0) = 0}$ $\quad$ 1 ćw.



tw. Taylora (Maclaurina)

$f(x) = R_n(x)$